A Hilbert state space model for the formation and dissolution of affinities among members in informal groups

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Abstract

This is the supplement of the paper with the same title which appeared in the proceedings of the workshop on The Problem Solving through the Applications of Mathematics to Human Behaviors. We shall add its motivation, tables and figures, and so on to the paper in order to complete the paper. Various solution curves in the figures suggest that our complex difference equation model covers enormous possible scenarios for the formation and dissolution of affinities among members of informal groups. Since the original model (Chino, 2002) assumes not only the *finite-dimensional Hilbert space* but also the *indefinite metric space* as the state space, this model might have wide applicability to all sort of phenomena in which asymmetric relationships among objects are essential. An extended version of our model is also proposed in this supplement, in which constant disturbance term is added. It is evident that this term enriches possible scenarios curiously and drastically. Differences between our model and the extant complex difference equation models are also discussed. Finally, we shall consider some open problems to be solved.

Key words: finite-dimensional Hilbert space, indefinite metric space, complex difference equation model, longitudinal asymmetric relational data matrices, n-body problem, Chino and Shiraiwa's theorem, tripartite deadlock, bifurcation theory, stability problem

1 Motivation

In any branch of science, we start with the careful observations of the phenomenon under study, and establish the so called *empirical law* which governs the phenomenon, first. For example, Kepler's law was established by Kepler using a body of astronomical observations of planetary motions which had been gathered by Tycho Brahe. According to this law, especially the first law, the Earth goes round the sun on the elliptical orbit.

However, the Kepler's law will never teach us why and how the Earth moves around the sun. Furthermore, we may ask whether the elliptical orbit is the only possible orbit or not. To answer these questions, we need two *theoretical laws.* One is Newton's second law, and the other Newton's law of gravitation.

On the one hand, Newton's second law, $F = ma$, is written as a second order differential equation, if we denote $x(t)$ as the position vector of the particle at time t:

$$\frac{dx^2(t)}{dt^2} = \frac{1}{m} F(x). \tag{1}$$

where m denotes the mass of the particle.

On the other hand, Newton's law of gravitation states that a body of mass m_1 exerts a force F on a body of mass m_2 such that $F = \frac{gm_1m_2}{r^2}$, where r denotes the distance between their centers of gravity and g a constant. Thus, if m_1 lies at the origin of R^3 and m_2 at $x \in R^3$, the force on m_2 can be written as

$$F(x) = -gm_1m_2\frac{x}{\mid x \mid^3}. \tag{2}$$

By solving the above differential equation, we can obtain possible three types of orbits, i.e., a *hyperbola*, *parabola*, and *ellipse* according to whether the total energy of the system $E > 0$, $E = 0$, or $E < 0$ (e.g., Hirsch & Smale, 1974, p.26). As is apparent from this example, an important role of any theoretical law is that it enables us to predict the phenomena which have never been observed empirically.

The above problem can easily be extended to the so-called "n-body problem". The Newtonian n-body problem, which is the prototype of other n-body problems, can generally be written as,

$$m_j \frac{dx_j^2(t)}{dt^2} = -grad_j\, V(x), \quad for \quad j = 1, 2, \cdots, n, \tag{3}$$

where $V(x)$ is a C^2 function, and

$$grad\, V = \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n} \right). \tag{4}$$

However, the n-body problem seems to be not easy to solve, and no clear picture has emerged on these equations (Hirsch & Smale, 1974).

In the next section we shall briefly discuss n-body problems mainly in social and behavioral sciences, and consider some basic problems associated with the n-body problem in these areas of research.

2 n-body problems mainly in social and behavioral sciences

Let us now ask some questions about n-body problems of the phenomena in social and behavioral sciences as well as biological sciences. The following list might be considered as typical examples of the phenomena observed in these sciences: relationships among members of any informal groups such as classmates at school or college and changes in these relationships over time, relationship called pecking order among a group of hens and cocks in a chicken house and changes in these relationships over time, amounts of trades among nations and their changes over time, strength of connections among neurons and changes in it over time. In all of these examples, members of the group can be thought of as constituents of the n bodies in the n-body problem.

What are the *state spaces* in which these objects are embedded? Are they embedded rationally in the *Euclidean space*? Have we already gathered ample observations necessary to construct some *empirical laws*? Can we construct some *theoretical laws* based on a few principles or statements about the phenomenon under study?

The first question is concerned with the basic problem in asymmetric multidimensional scaling (abbreviated hereafter, as *asymmetric MDS*) in psychometrics (e.g., Chino, 2012), and the answer to the second question is negative in the strict sense, because the distances from member j to member k as well as the one from member k to member j defined in the Euclidean space are the same and this relation contradicts the asymmetric similarities between members usually observed in the phenomena in social and behavioral sciences. To cope with this problem, various augmented distance models have been proposed which assume the Euclidean space or Minkowski's l-metric space and augment the distance by adding some factors on asymmetry (e.g., Okada & Imaizumi, 1984, 1987).

In contrast, Chino and Shiraiwa (1993) have shown that objects can be embedded in a finite dimensional Hilbert space or an indefinite metric space, given an asymmetric similarity matrix $\boldsymbol{S}$ whose (j, k) element is the observed similarity from object j to object k. Let us here construct a *Hermitian matrix* $\boldsymbol{H}$ as follows:

$$\boldsymbol{H} = (\boldsymbol{S} + \boldsymbol{S}^t)/2 + i\,(\boldsymbol{S} - \boldsymbol{S}^t)/2. \tag{5}$$

where $i^2 = -1$. Chino and Shiraiwa have proven that a necessary and sufficient condition for a set of distances $d_{jk} = d_{kj}$ to be the true interpoint distances in a (complex) Hilbert space is the *positive semi-definiteness* of $\boldsymbol{H}$. This is an extension of the *Schoenberg-Young-Householder theorem* (Chino et al., 2012) on MDS to the case of the complex space.

Chino and Shiraiwa have also shown that objects are embedded in an appropriate space by solving the eigenvalue problem of $\boldsymbol{H}$. In fact we have

$$\boldsymbol{H} = \boldsymbol{X}\boldsymbol{\Omega}_s\boldsymbol{X}^t + i\,\boldsymbol{X}\boldsymbol{\Omega}_{sk}\boldsymbol{X}^t, \tag{6}$$

where

$$\boldsymbol{\Omega}_s = \begin{pmatrix} \boldsymbol{\Lambda}, & \boldsymbol{O} \\ \boldsymbol{O}, & \boldsymbol{\Lambda} \end{pmatrix}, \quad \boldsymbol{\Omega}_{sk} = \begin{pmatrix} \boldsymbol{O}, & -\boldsymbol{\Lambda} \\ \boldsymbol{\Lambda}, & \boldsymbol{O} \end{pmatrix}, \tag{7}$$

and $\boldsymbol{\Lambda}$ is a diagonal matrix of order n which is the number of non-zero eigenvalues of $\boldsymbol{H}$, i.e., $\boldsymbol{\Lambda} = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$. The matrix $\boldsymbol{X}$ is the special *real* $N \times 2n$ *coordinate matrix* of objects, i.e., $\boldsymbol{X} = (\boldsymbol{U}_r, \boldsymbol{U}_c)$, where $\boldsymbol{U}_1 = \boldsymbol{U}_r + i\,\boldsymbol{U}_c$ and $\boldsymbol{U}_1$ are composed of the *complex* eigenvectors of $\boldsymbol{H}$ corresponding to its non-zero eigenvalues. It should be noticed that all the eigenvalues of $\boldsymbol{H}$ is *real.*

It is easy to show that Eq. (6) can be rewritten as

$$\boldsymbol{S} = \boldsymbol{X}\boldsymbol{\Omega}_s\boldsymbol{X}^t + \boldsymbol{X}\boldsymbol{\Omega}_{sk}\boldsymbol{X}^t. \tag{8}$$

This equation shows the relation between observed similarities among objects and the coordinates of objects in an appropriate complex space. For further detail, see elsewhere (Chino and Shiraiwa, 1993; Chino, 2012).

Table 1: Sociometric data for 10 students in a high school

rater\ratee	1	2	3	4	5	6	7	8	9	10
1	4	3	4	3	5	5	6	4	7	7
2	2	3	4	6	7	6	5	5	4	5
3	4	4	3	5	5	4	4	3	4	5
4	4	7	6	3	7	7	4	6	4	5
5	1	7	6	7	4	7	6	6	6	5
6	4	5	4	6	5	7	4	4	4	4
7	4	5	4	4	3	3	6	6	4	6
8	2	4	4	4	5	4	5	2	4	4
9	6	5	5	5	5	6	5	5	4	6
10	4	4	4	3	4	4	4	4	4	4

(reproduced from Chino (1978))

If $\boldsymbol{S}$ are measured at a *ratio scale* level, we can embed objects in an appropriate complex space. Chino and Shiraiwa (1993) call the above method for asymmetric MDS *Hermitian Form Model* (abbreviated as HFM). In application, however, it is often the case that $\boldsymbol{S}$ are not measured at a ratio scale level. For example, Table 1 shows a sociometric data, in which each element designates the affinity of a member toward another member in a class of a senior-high school. It is measured at an *interval level.* In such a case we must estimate the coordinates of objects by a suitable method. Saburi and Chino (2008) show one such method. In their method $\boldsymbol{S}$ may be measured at either the ordinal, interval, or ratio level.

As regards the third question, we have not had ample observations regarding many of the problems listed above, especially in cases in social and behavioral sciences. In such a situation, a set of longitudinal asymmetric relational data matrices (to be precise, longitudinal attraction data matrices) among members of a dormitory in The University of Michigan is a rare example, which was gathered by Newcomb (1961). In general, it is laborious to gather such a longitudinal asymmetric data matrices in phenomena in the social and behavioral sciences. Thus, it will be appropriate and natural at present to simulate such phenomena using some mathematical models.

One such candidate may be a *difference equation model.* Considering the Chino and Shiraiwa's theorem, we assume that the state space in which objects or members are embedded is a Hilbert space or an indefinite metric space. Furthermore, we assume that members obey the following basic principles of interpersonal behaviors:

1. The asymmetric sentiment relationships among members make their affinities change.
2. If a member has a positive sentiment toward another member, then he or she approaches to the target member.
3. If a member has a negative sentiment toward another member, then he or she aparts from the target member.

There exist two minor principles in this difference equation model, as listed below:

1. The magnitude of change in coordinate of members is proportional to the *sine* of the difference in angles (arguments) between two members in a complex plane.
2. The magnitude of change in coordinate of members is proportional to the *norm* of the coordinate in a complex plane.

The first one is concerned with our asymmetric MDS, HFM, and the magnitudes of change in coordinates of members take the maximum value when the angle is $\pm\pi/2$. The second one is associated with the *self-similarity* of each member, because the norm represents the magnitude of similarity to oneself.

One of the models which fulfill these requirements is the finite-dimensional complex difference equation model proposed by Chino (2002) :

$$\boldsymbol{z}_{j,n+1} = \boldsymbol{z}_{j,n} + \sum_{m=1}^{q} \sum_{k \neq j}^{N} \boldsymbol{D}_{jk,n}^{(m)} \boldsymbol{f}^{(m)}(\boldsymbol{z}_{k,n} - \boldsymbol{z}_{j,n}), \quad j = 1, 2, \cdots, N. \tag{9}$$

Here,

$$\boldsymbol{f}^{(m)}(\boldsymbol{z}_{k,n} - \boldsymbol{z}_{j,n}) = \left((z_{k,n}^{(1)} - z_{j,n}^{(1)})^m, (z_{k,n}^{(2)} - z_{j,n}^{(2)})^m, \cdots, (z_{k,n}^{(p)} - z_{j,n}^{(p)})^m \right)^t. \tag{10}$$

Moreover,

$$\boldsymbol{D}_{jk,n}^{(m)} = diag \left(w_{jk,n}^{(1,m)}, w_{jk,n}^{(2,m)}, \cdots, w_{jk,n}^{(p,m)} \right), \tag{11}$$

$$w_{jk,n}^{(l,m)} = a_n^{(l,m)} r_{j,n}^{(l,m)} r_{k,n}^{(l,m)} \sin \left(\theta_{k,n}^{(l,m)} - \theta_{j,n}^{(l,m)} \right), \quad l = 1, 2, \cdots, p, \quad m = 1, 2, \cdots, q. \tag{12}$$

In the above model, N denotes the number of members in informal groups, and p and q represent appropriate constants. Moreover, $r_{j,n}$ and $\theta_{j,n}$ denote, respectively,the norm and the argument of member j's position vector, $\boldsymbol{z}_{j,n}$, at time n. Of course, the vector is a p-dimensional vector in a finite-dimensional complex space. The space may be either a Hilbert space or an indefinite metric space. In this paper we assume the former space.

On the one hand, it is assumed in Eq. (9) that the *positive direction of the configuration of members* in each of the complex planes associated with the complex dimensions is *counter-clockwise* (Chino, 1978, 1990). On the other hand, it is assumed that the positive direction is *clockwise* in HFM (Chino & Shiraiwa, 1993). In this paper we choose the clockwise direction in each of the complex plane as the positive direction. As a result, Eq. (9) must be rewritten as,

$$\boldsymbol{z}_{j,n+1} = \boldsymbol{z}_{j,n} + \sum_{m=1}^{q} \sum_{k \neq j}^{N} \boldsymbol{D}_{jk,n}^{(m)} \boldsymbol{f}^{(m)}(\boldsymbol{z}_{j,n} - \boldsymbol{z}_{k,n}), \quad j = 1, 2, \cdots, N. \tag{13}$$

Here, it is necessary to change the signs of each of the elements in Eq. (10) by exchanging j and k. Moreover, we simplified the factors in the sine function of Eq. (12) a bit, and added new parameters b and c, as follows:

$$a_n^{(l,m)} r_{j,n}^{(l,m)} r_{k,n}^{(l,m)} = b(r_{j,n}^{(l,m)} r_{k,n}^{(l,m)})^c. \tag{14}$$

In this supplement, we also discuss a more general model, in which a small complex constant (vector) is added to Eq. (13), i.e.,

$$\boldsymbol{z}_{j,n+1} = \boldsymbol{z}_{j,n} + \sum_{m=1}^{q} \sum_{k \neq j}^{N} \boldsymbol{D}_{jk,n}^{(m)} \boldsymbol{f}^{(m)}(\boldsymbol{z}_{j,n} - \boldsymbol{z}_{k,n}) + \boldsymbol{z}_0, \quad j = 1, 2, \cdots, N. \tag{15}$$

In the appendix, we show some restricted simulation results of the above general difference equation model described by Eq. (15). It will be seen that the above constant enriches possible scenarios for the formation and dissolution of affinities among members of (informal) groups.

3 Restricted simulation results of a general complex difference equation model

In this section and the appendix, we run several simulations on a computer to examine our complex difference equation models under different conditions. In all of the simulations, we first show the initial configuration of objects (members) which we assumed. In this paper, we assume the unidimensional Hilbert space to be the state space, i.e., the usual complex plane, although our model assumes a finite-dimensional Hilbert space or an indefinite metric space in general.

In looking at configurations of objects in these simulations, some cautions must be exercised. In psychometrics it is usual to depict the positions of objects as dots. However, in this section we draw them as the position vectors in order to show the change in these positions over time (iteration) as clearly as possible. Another reason is that in HFM similarity between two objects is represented as an inner product (to be precise, Hermitian inner product), the origin is crucial. It should be noticed that the positive direction of the configuration is *clockwise* since we utilize HFM in this simulation.

Simulation 1: dyad relation, N=2, n=200, p=q=1, b=c=1/50

In simulation 1, we examine change in a dyad relation during 200 iterations, where the number of members equals 2, p=q=1, and b=c=1/50 in Eq. (13). Fig. 1 illustrates the change in configurations of two members in the complex plane. It is apparent that the skewness of affinity between the two gradually decreases, as both of them rotate in the clockwise direction. Here, it should be noticed from Fig.2-a, 2-b that self-similarities of both members, i.e., norms of

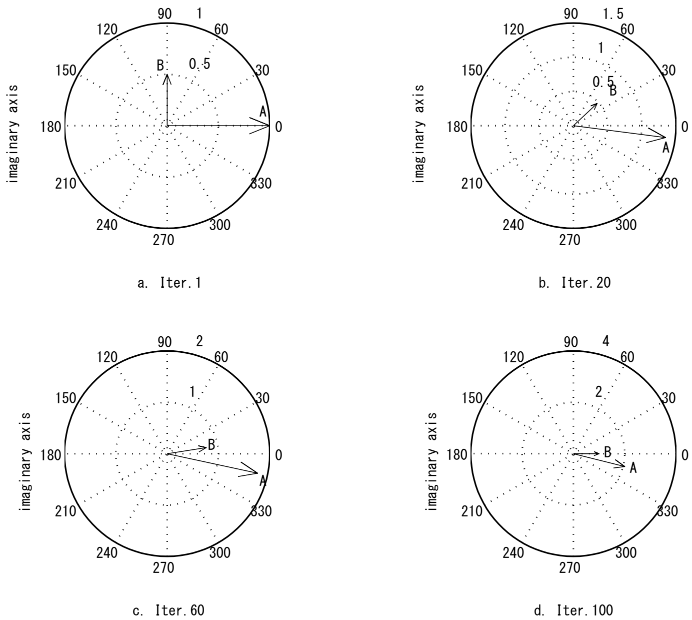

Figure 1: Locomotion of a dyad relation in a unidimensional Hilbelt space

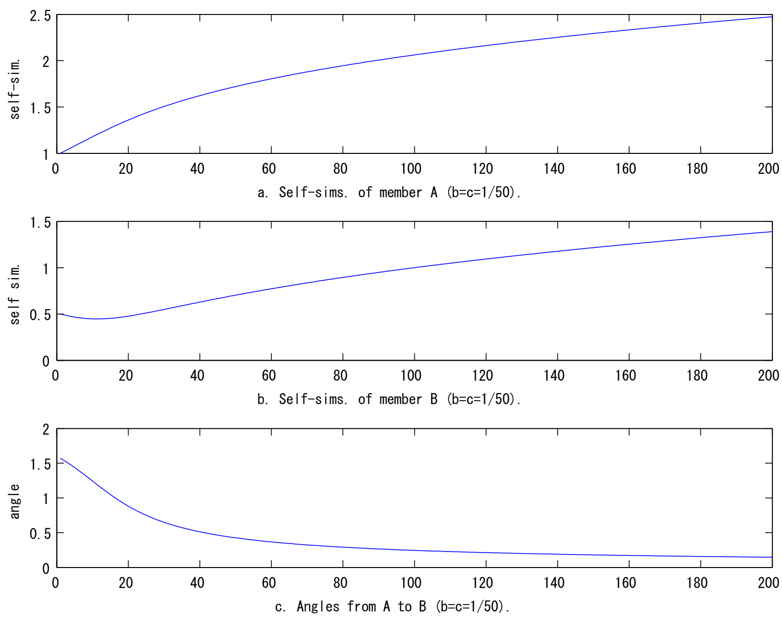

Figure 2: Changes in self-similarities of the dyad (2-a, b) and change in the angle (2-c)

their position vectors, increase monotonically after earlier iterations. In contrast, the skewness of affinity between them decreases monotonically immediately after the first iteration, as is apparent from Fig.2-c.

Finally, Fig.3 depicts orbits of the dyad during 200 iterations in the same complex plane. In this figure, a black dot represents the origin. It is apparent from this figure that both of the two members apart from the origin as iteration proceeds, moving on the same line while preserving the distance between them.

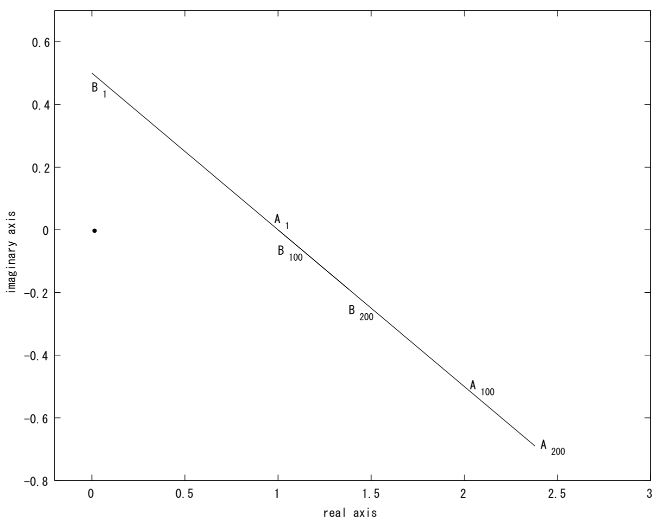

Figure 3: Orbits of the dyad in simulation 1

As can be seen from simulation 1, locomotion of dyad in the complex plane is simple and *linear*, and the skewness of affinity between the dyad diminishes asymptotically. However, in the case of triad relation, their locomotions become *nonlinear* and show curious scenarios. However, in many cases the skewness of affinity among members diminishes asymptotically after a fairly long iterations. We shall show such scenarios in the following several subsections.

4 Discussion

In this paper, we have first made a brief review of the asymmetric MDS methods which have been developed in Psychometrics, and have pointed out that those methods which deal with longitudinal asymmetric relational data are rare. Next, we have made a restricted simulation study, in which a revised version of a complex difference equation model proposed first by Chino (2002) is examined. In this paper, we have examined a special difference equation model whose state space is assumed to be a unidimensional Hilbert space.

As shown in the simulation, it was found that this model includes a body of curious scenarios concerning the change in affinities of members. Furthermore, we have proposed a more general model which add a complex constant (vector) in the above model. This model extend the original one such that it introduces interesting oscillation phenomena in the process of change in affinities over time (iteration). Psychological implications are discussed.

There are a lot of open problems to be discussed. Some of them are (1) the problem of estimating model parameters, given empirical data, (2) identification of the bifurcation parameters by for example utilizing SEM, (3) stability analysis of our model, (4) the problem of defining energy in our model.

In the appendix, we have shown major results on several simulation studies other than Simulation 1.

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Appendix

Simulation 2: triad relation, N=3, n=7000, p=q=1, b=c=1/50

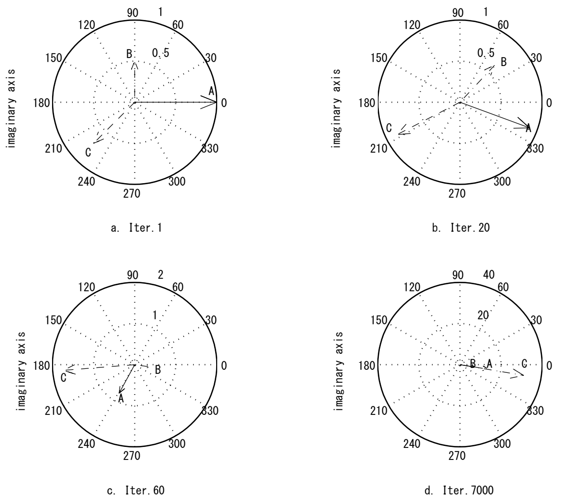

Figure 4: Locomotion of a triad relation in a unidimensional Hilbelt space

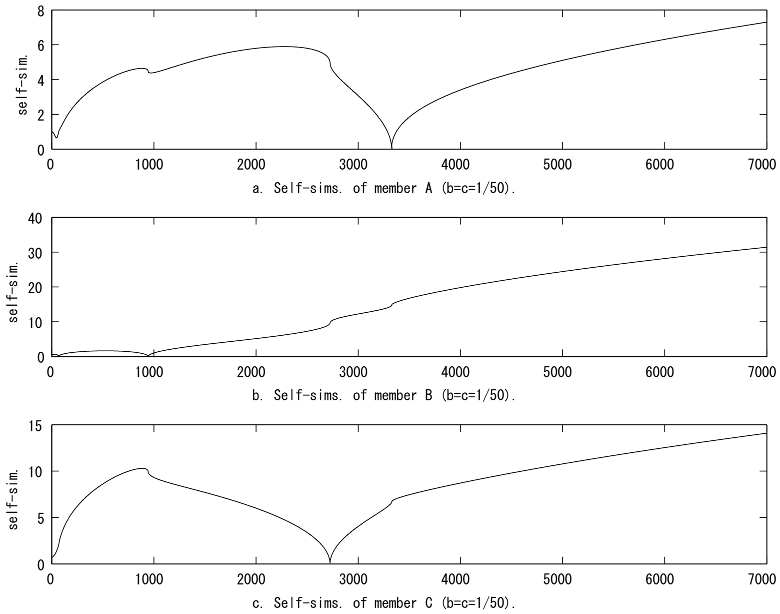

Figure 5: Changes in self-similarities of the triad

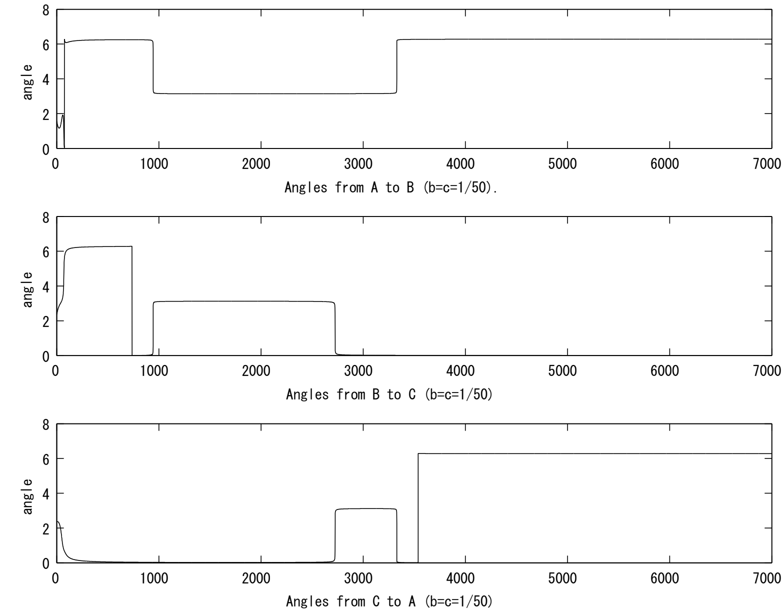

Figure 6: Changes in angles among the triad

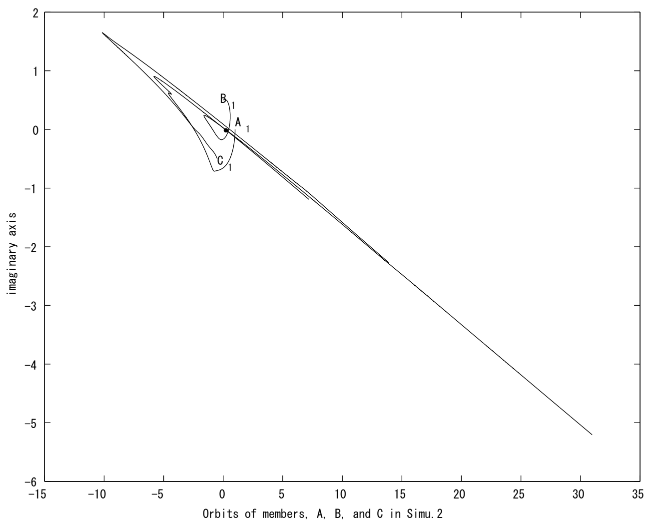

Figure 7: Orbits of the triad in simulation 2

Simulation 3: tripartite deadlock, N=3, n=600, p=q=1, b=c=1/50

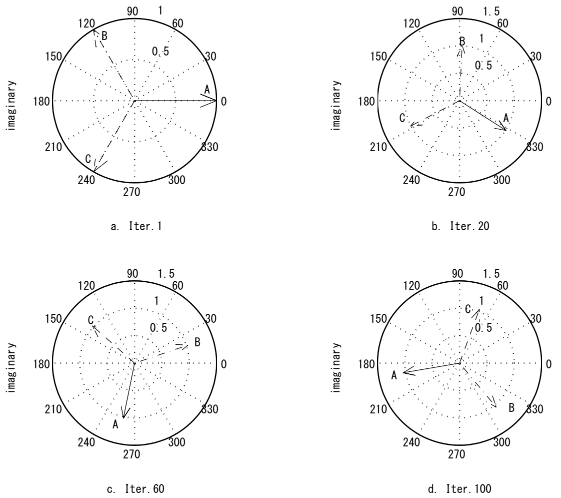

Figure 8: Locomotion of a tripartite in a unidimensional Hilbelt space

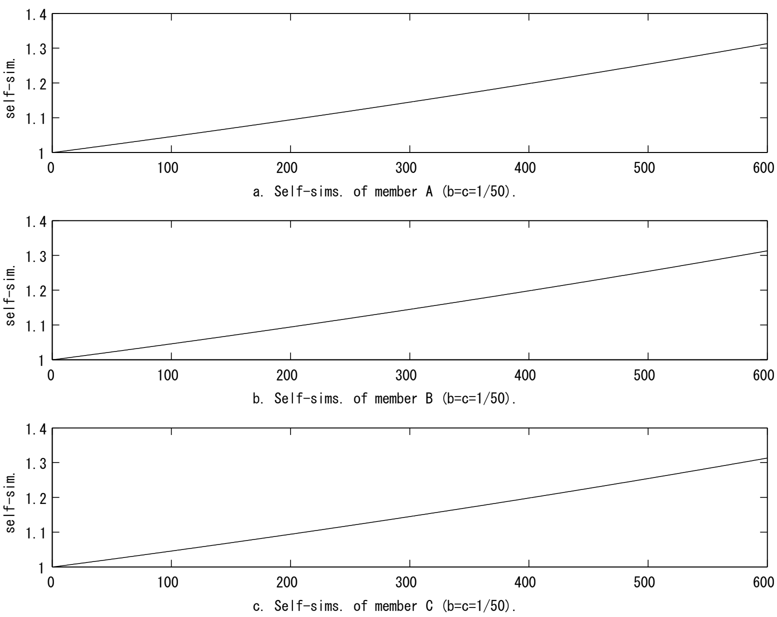

Figure 9: Changes in self-similarities of the tripartite

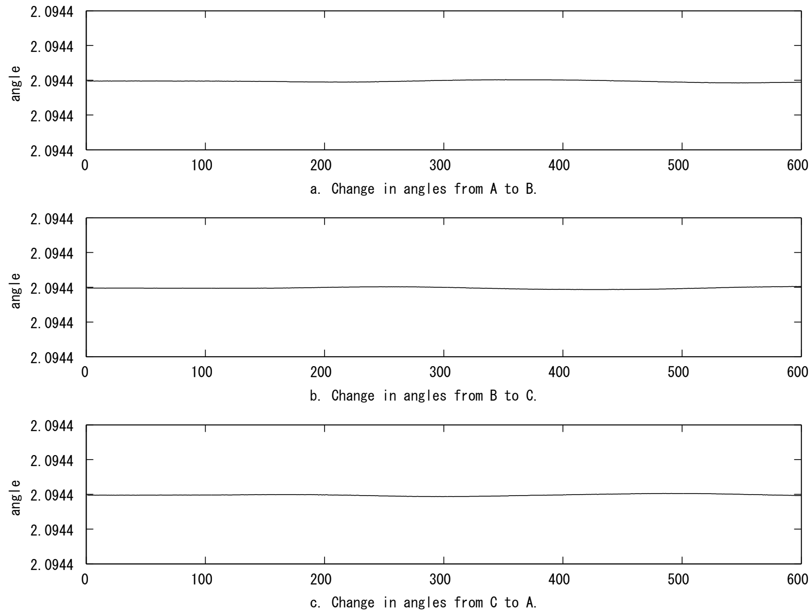

Figure 10: Changes in angles among the tripartite

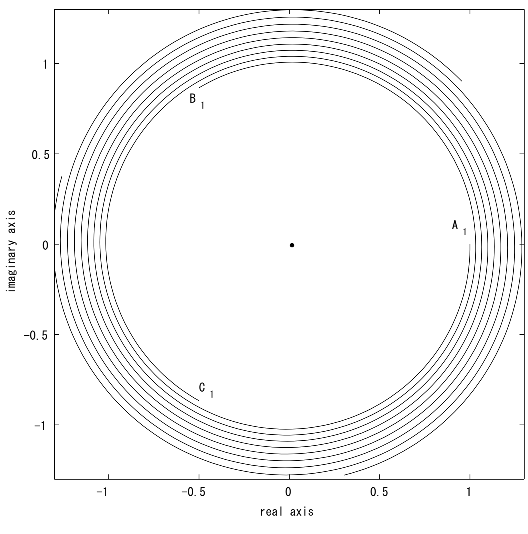

Figure 11: Orbits of the tripartite in simulation 3

Simulation 4: linear triad relation, N=3, n=200, p=q=1, b=c=1/50

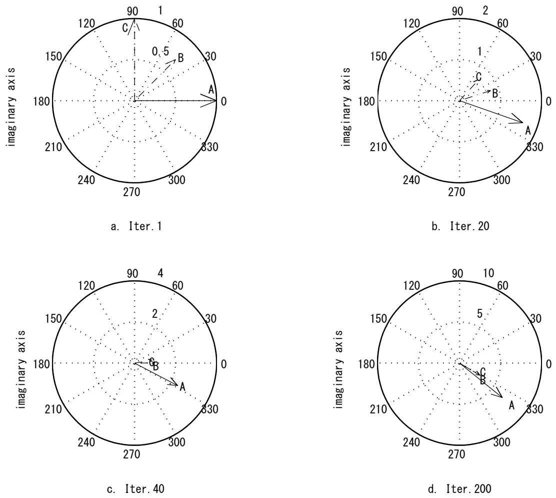

Figure 12: Locomotion of a triad in a unidimensional Hilbelt space

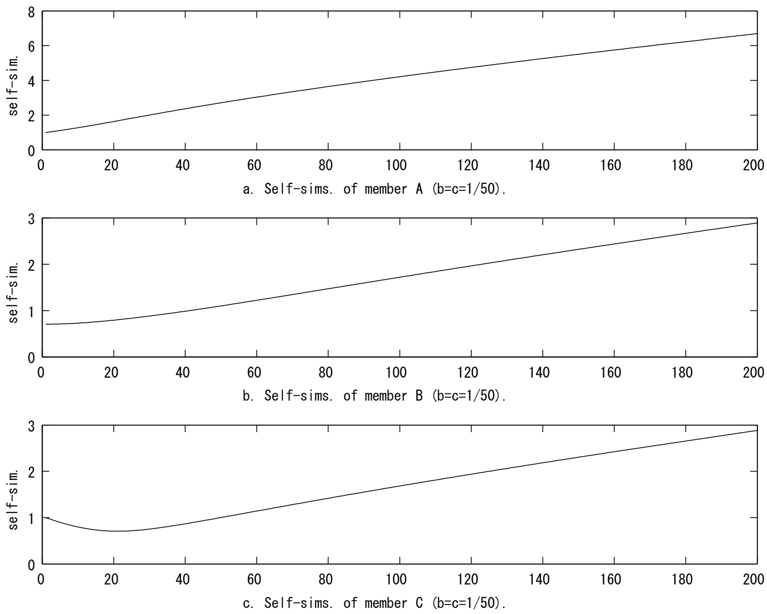

Figure 13: Changes in self-similarities of the triad

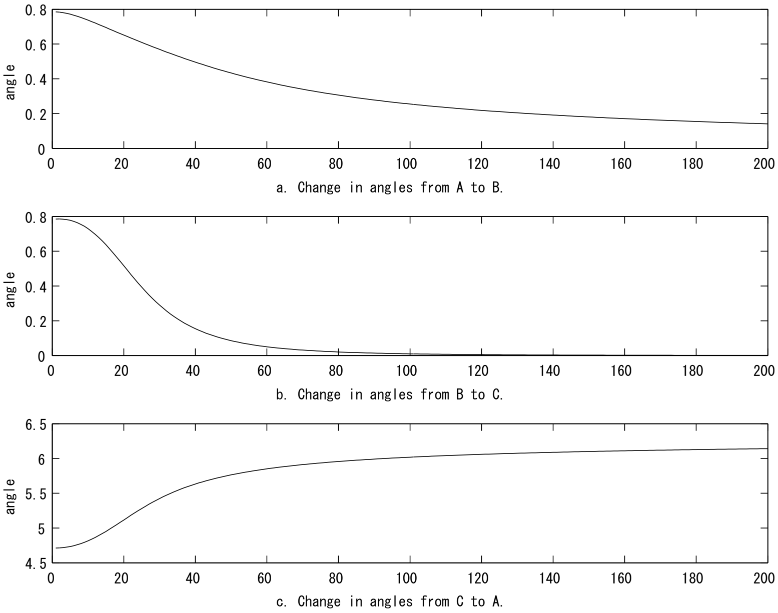

Figure 14: Changes in angles among the triad

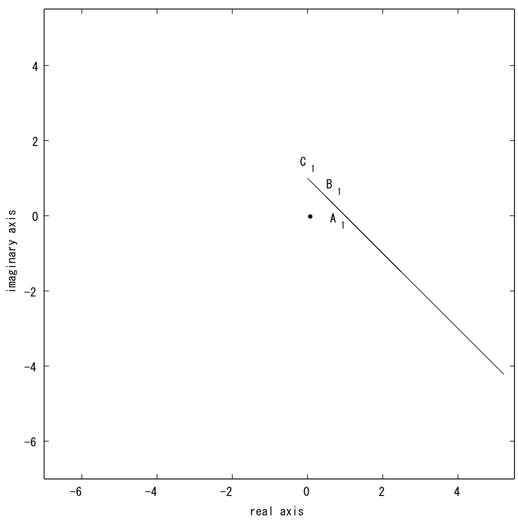

Figure 15: Orbits of the triad in simulation 4

Simulation 5: dyadic relation, N=2, n=200, p=1, q=2, b=c=1/50

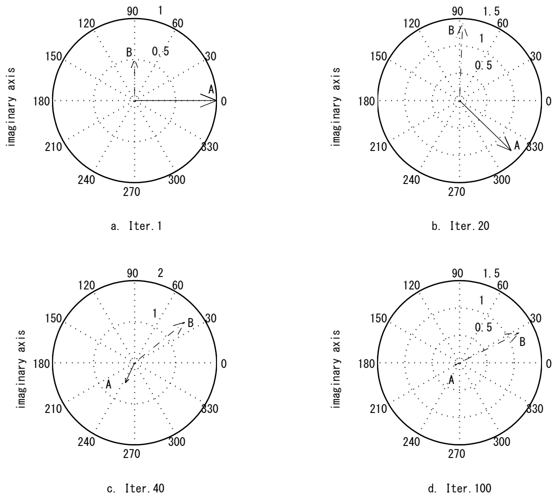

Figure 16: Locomotion of a dyad in a unidimensional Hilbelt space

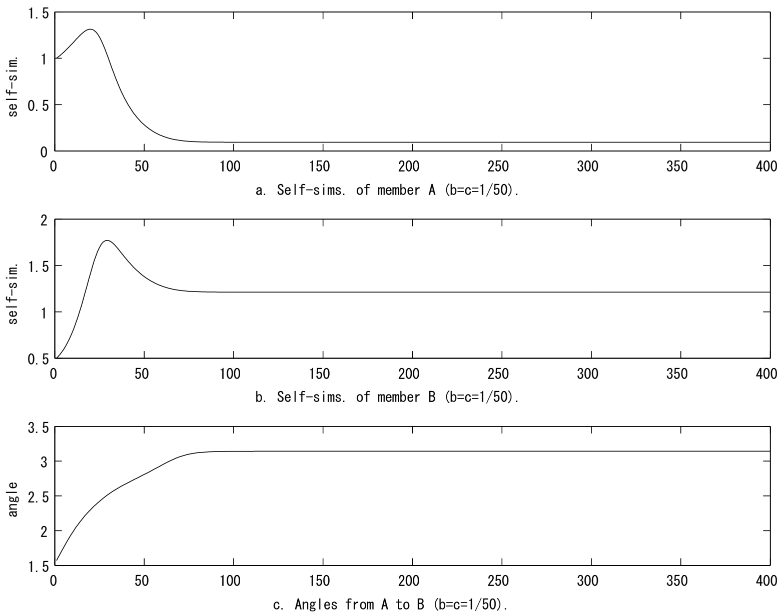

Figure 17: Changes in self-similarities of the dyad (17-a, b) and change in the angle (17-c)

It should be noticed that the initial configuration in this simulation is the same as that in simulation 1. Nevertheless, the configuration at iteration 100 is completely different from that, as shown in Fig.16. Moreover, self-similarities of the triad *no longer increase monotonically* after after earlier iteration, but approach to some constants asymptotically, as depicted in Fig.17. Furthermore, orbits of the dyad are *no longer linear* as shown in Fig.18.

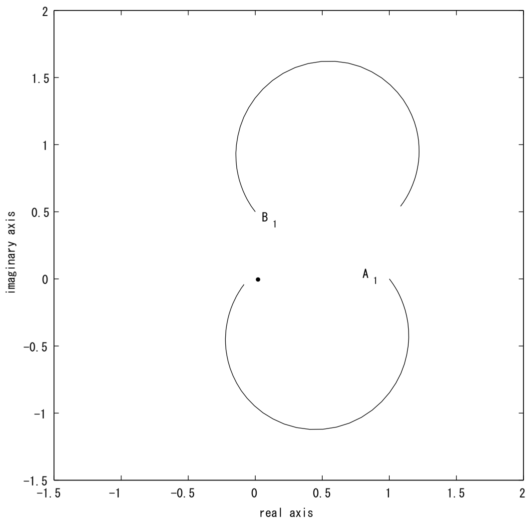

Figure 18: Orbits of the dyad in simulation 5

Simulation 6: tripartite deadlock, N=3, n=600, p=1, q=2, b=c=1/50

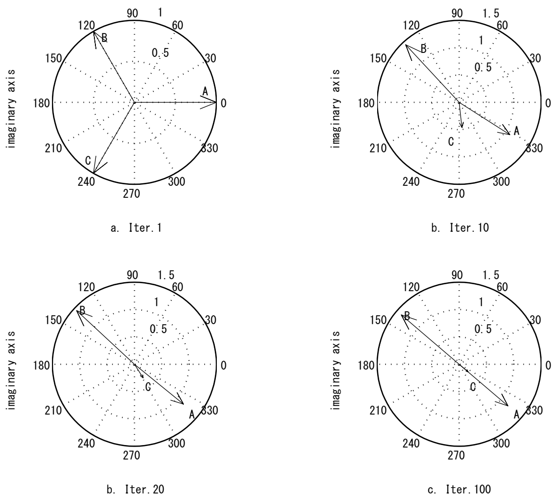

Figure 19: Locomotion of a tripartite in a unidimensional Hilbelt space

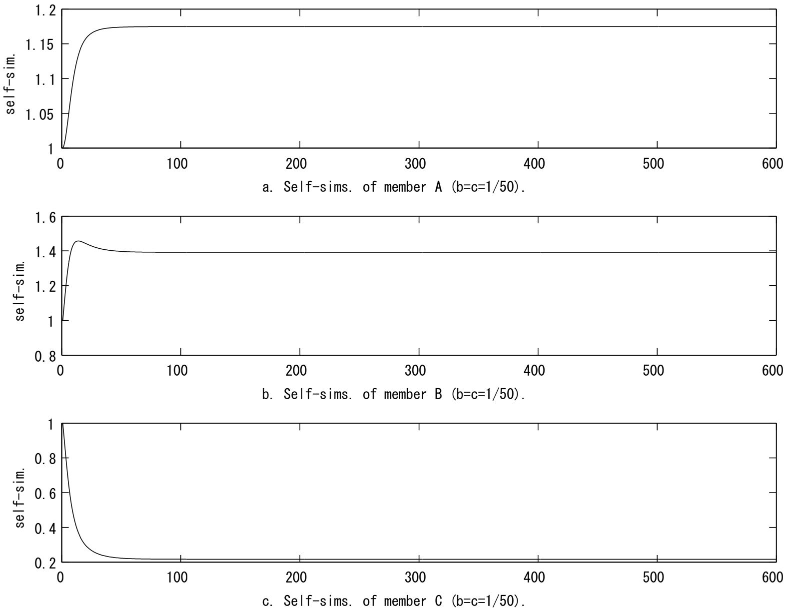

Figure 20: Changes in self-similarities of the tripartite

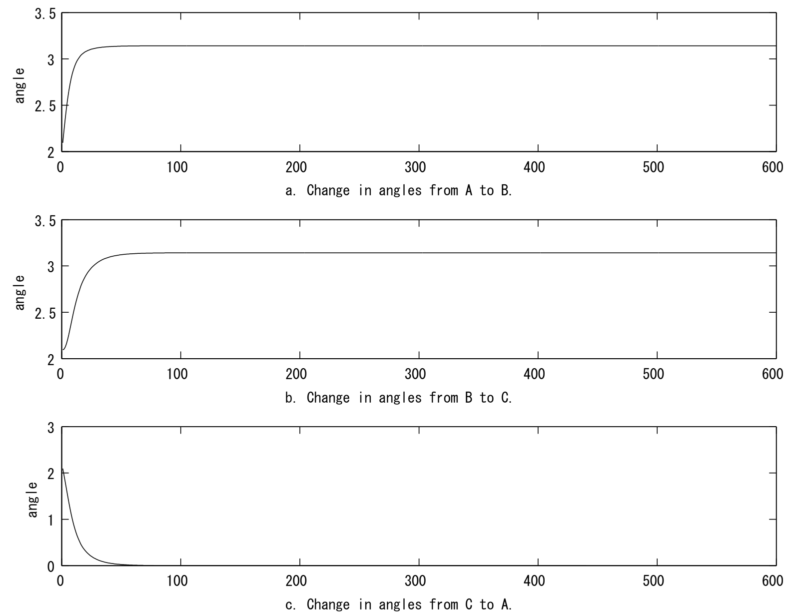

Figure 21: Changes in angles among the tripartite

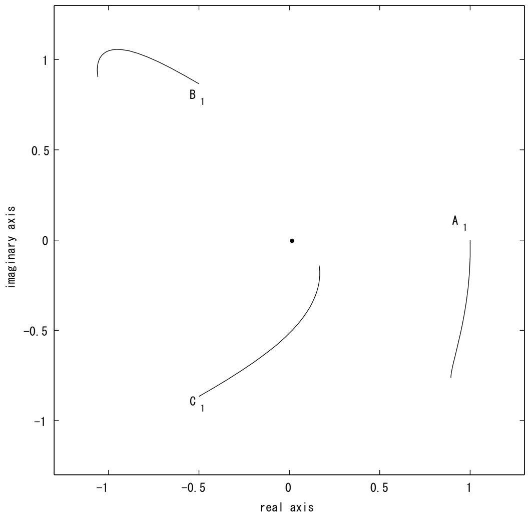

Figure 22: Orbits of the tripartite in simulation 6

Simulation 7: dyad relation, const=0.01i, N=2, n=1000, p=q=1, b=c=1/50

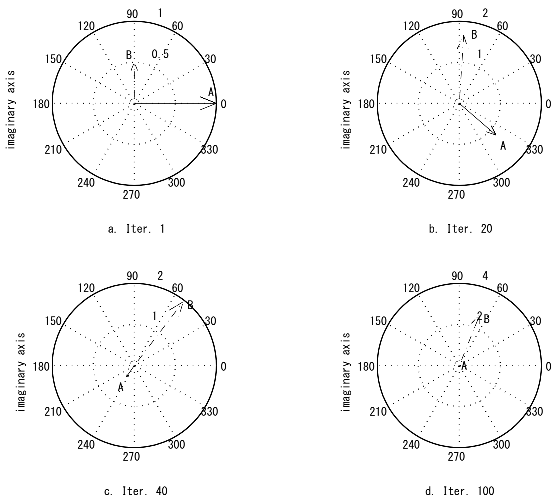

Figure 23: Locomotion of a dyad relation in a unidimensional Hilbelt space

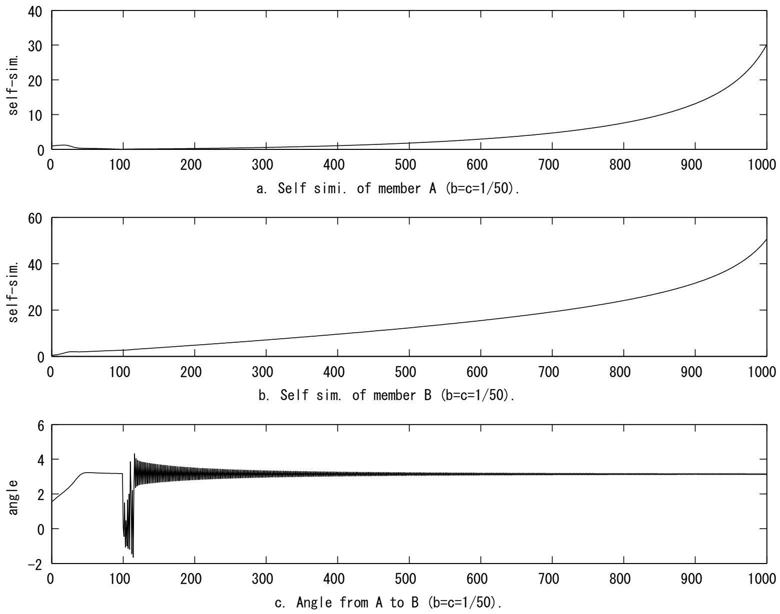

Figure 24: Changes in self-similarities of the dyad (2-a, b) and change in the angle (2-c)

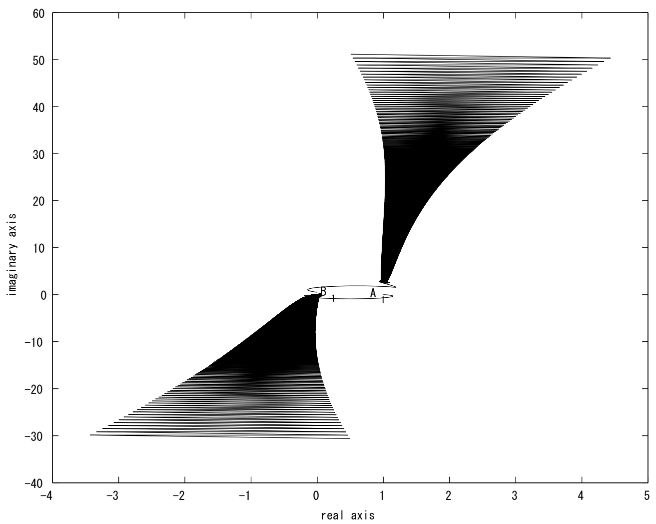

Figure 25: Orbits of the dyad in simulation 7

Simulation 8: triad relation, const=0.0001i, N=3, n=20000, p=q=1, b=c=1/50

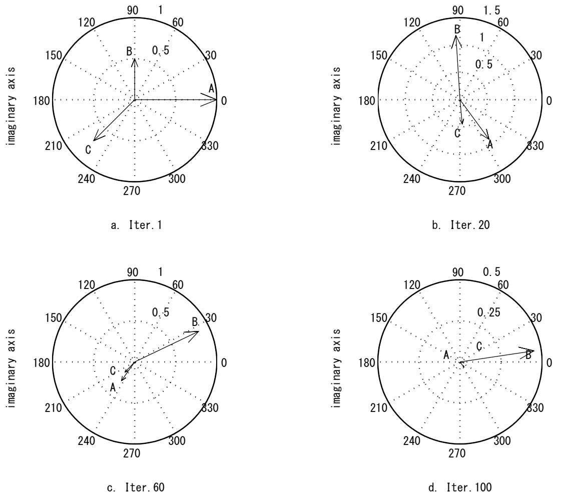

Figure 26: Locomotion of a triad relation in a unidimensional Hilbelt space

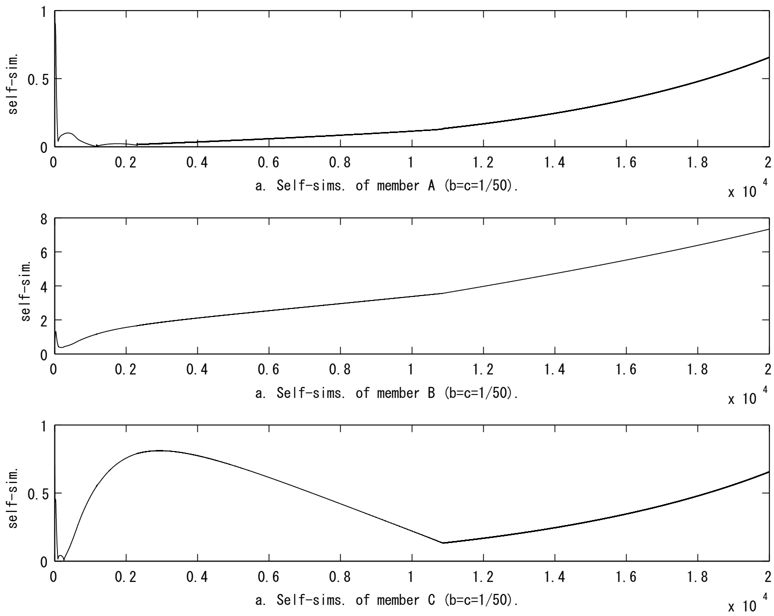

Figure 27: Changes in self-similarities of the triad

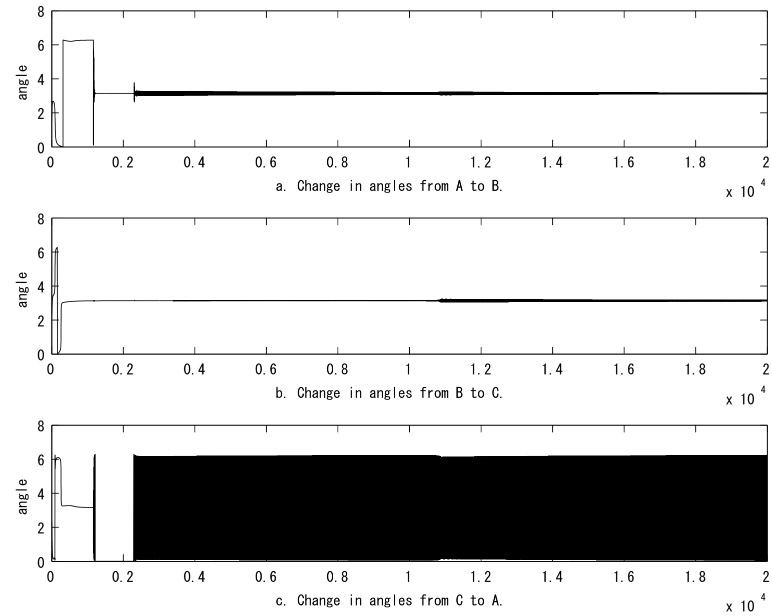

Figure 28: Changes in angles among the triad

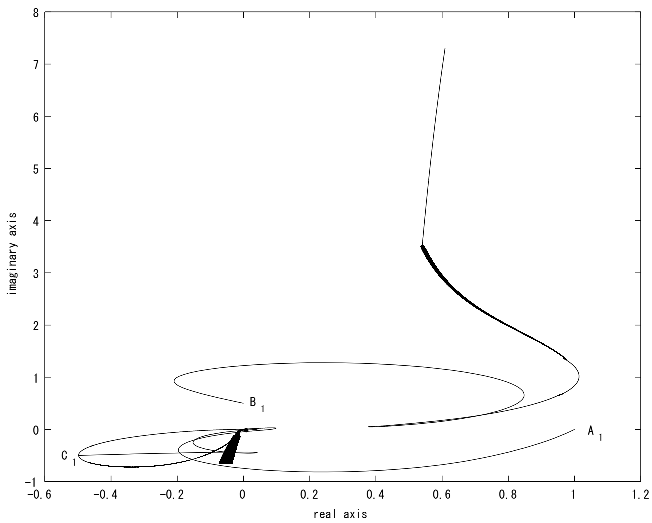

Figure 29: Orbits of the triad in simulation 8

Simulation 9: triad relation, const=0.0001i, N=3, n=8600, p=1, q=2, b=c=1/50

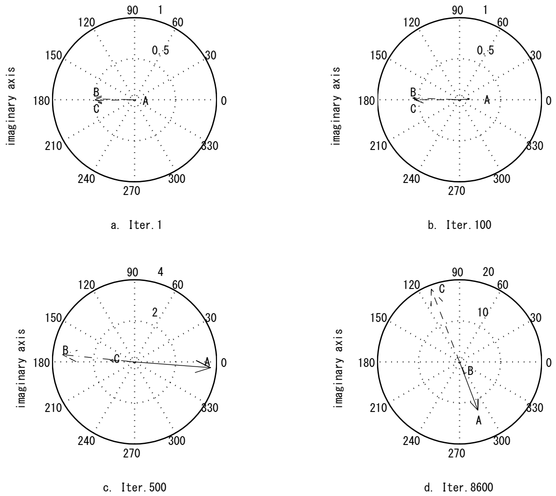

Figure 30: Locomotion of a triad relation in a unidimensional Hilbelt space

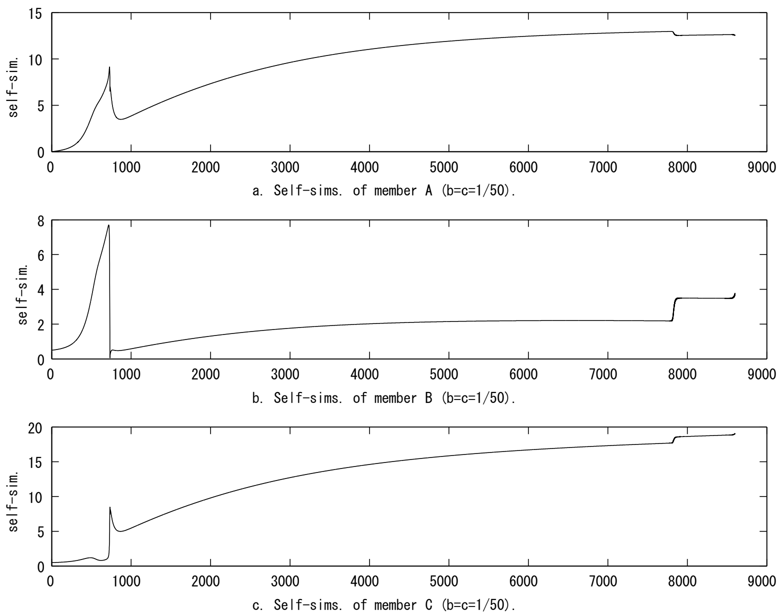

Figure 31: Changes in self-similarities of the triad

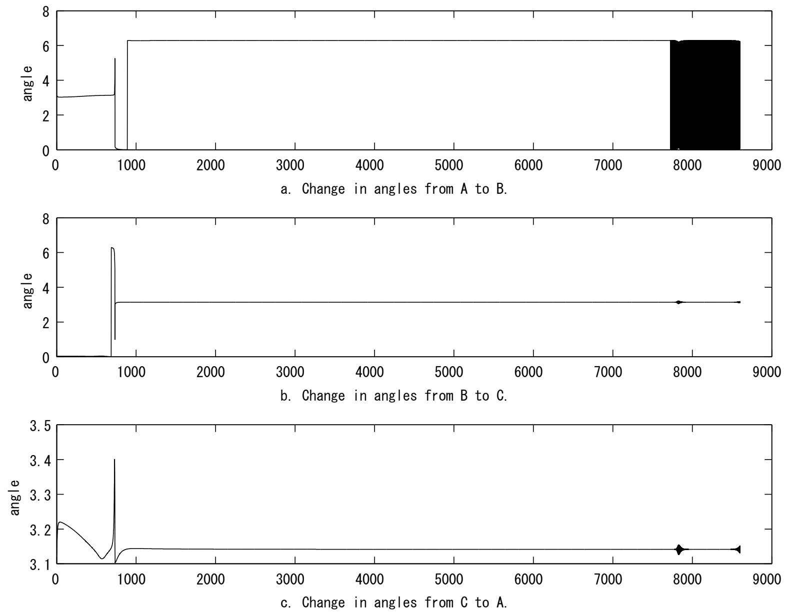

Figure 32: Changes in angles among the triad

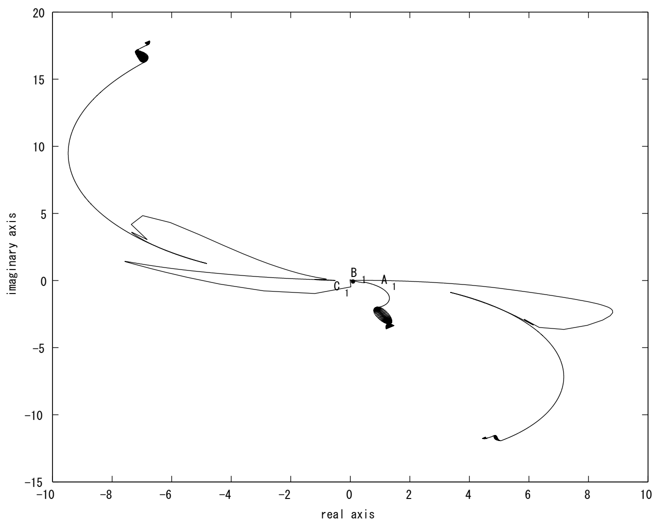

Figure 33: Orbits of the triad in simulation 9