Asymmetric multidimensional scaling and related topics

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Acknowledgement

Firstable, let me express my gratitude to Prof. Spokoiny and Prof. Mucha for inviting me to this research seminar. It is of great honor for me to have a talk at this institute which is dedicated to Dr. Karl Weierstrass who is known as a great mathematician in the world.

Organization of my talk

Today, I will talk about the developments in the past and future possibilities of the asymmetric multidimensional scaling, abbreviated hereafter as asymmetric MDS. The organization of my talk is as follows:

1. **Introduction to symmetric and asymmetric MDS**

   First, I will briefly review the body of traditional work on asymmetric MDS since Young (1975) proposed the ASYMSCAL, after a brief look at the theory and an application of the symmetric MDS.

   Then, I will introduce briefly some of the asymmetric MDS developed recently. These include extensions of the traditional two-way asymmetric MDS to the three-way MDS, versions of the traditional methods, and so on. For some of these details, see Cox and Cox (2001).

2. **Possible developments in the future**

   Next, I will show you three possible developments in the future.

   - **Developments of some inferential methods and statistics**
     One is to develop methods which enable the use of inferential statistics such as those for testing various symmetry hypotheses about the data under study.
   - **Linkage to the analysis of contingency tables**
     Another may be to consider the linkage of the asymmetric MDS to the analysis of contingency tables. This enables to have its linkage further, to the models developed in cognitive psychology. Although Zielman & Heiser (1996) and De Rooij & Heiser (2005) have made an inroad regarding this problem, their method is a mixture of the analysis of contingency tables and the symmetric MDS, and is designed basically for the count data, especially contingency tables.
   - **Linkage to the dynamical system theories**
     The other may be to make a linkage to the dynamical system theories in mathematics. Chino & Nakagawa (1990) have made an inroad regarding this problem. In order to develop this problem further, a Hilbert space theorem proven by Chino & Shiraiwa (1993) may be useful. Some preliminary approaches to these problems will be shown.

1 Introduction

Let me begin this lecture with a narrow **definition of multidimensional scaling** by Cox and Cox (2001). According to them, it is the search for a low dimensional space, usually Euclidean, in which points in the space represent the objects, one point representing one object, and such that the distance between the points in the space, match, as well as possible, the original dissimilarities between the objects.

Asymmetric MDS is an extension of the traditional MDS to the asymmetric relational data. Since the traditional MDS originated by Torgerson (1952) deals with the symmetric relational data, I shall call it the **symmetric MDS**.

A famous example of the application of the symmetric MDS is a set of flying mileages between 10 U.S. cities which appears in Kruskal and Wish (1978).

```
*--------------------------------------------------------------------------*
|                                                          March 28, 2006 |
   program name: classic-mds.sas
|                                                                         |
     This program is a sas program for performing a classical mds to the
   flying mileages between 10 us cities, which is shown in Kruskal and Wish
   (1978, pp.7-9).  It appears in chapter 40 of the SAS online documents.
|                                                                         |
*--------------------------------------------------------------------------*;
data   city;
      title 'Analysis of Flying Mileages Between Ten U.S. Cities';
      input  (atlanta chicago denver houston losangeles
              miami newyork sanfran seattle washdc) (5.)
              @56 city $15.;
      datalines;
    0                                                 Atlanta
  587    0                                            Chicago
 1212  920    0                                       Denver
  701  940  879    0                                  Houston
 1936 1745  831 1374    0                             Los Angeles
  604 1188 1726  968 2339    0                        Miami
  748  713 1631 1420 2451 1092    0                   New York
 2139 1858  949 1645  347 2594 2571    0              San Francisco
 2182 1737 1021 1891  959 2734 2408  678    0         Seattle
  543  597 1494 1220 2300  923  205 2442 2329    0    Washington D.C.
   ;
proc mds data=city level=absolute out=out;
      id city;
run;

%plotit(data=out, datatype=mds, labelvar=city,
            vtoh=1.75, labfont=swissb);
run;
```

Figure 1: A sas program for executing the MDS procedure

Given this set of data, say, the MDS procedure of SAS recovers the map of the cities as follows (Fig.2). This means that the locations of objects are intelligible if we know the mutual distances between them in Euclidean space. The reason for this is based on the famous **Young-Householder theorem** (Young & Householder, 1938). <Fig. 2

According to this theorem, a necessary and sufficient condition for a set of numbers $d_{ij} = d_{ji}$ to be the mutual distances of a real set of points in Euclidean space is that the inner product matrix $\boldsymbol{B}$ whose elements b_{ij} are define by

$$b_{ij} = \frac{1}{2}\,(d_{in}^2 + d_{jn}^2 - d_{ij}^2), \tag{1}$$

be p.s.d., and in this case the set of points is unique apart from a Euclidean transformation. Here, the origin is at point n.

For fallible data, it is usual that the origin is placed at the centroid of all the objects. As a result, we revise the equation (1) as follows:

$$b_{ij}^* = \frac{1}{2}\left[\frac{1}{n}\sum_{i}^{n} d_{ij}^2 + \frac{1}{n}\sum_{j}^{n} d_{ij}^2 - \frac{1}{n^2}\sum_{i}^{n}\sum_{j}^{n} -d_{ij}^2\right]. \tag{2}$$

Methods of fitting a lower dimensional set of points to a given set are also available by utilizing the Eckart-Young theorem (Eckart & Young, 1936):

The symmetric MDS has been extended by many researchers since the work of Torgerson, and has been applied to relational data in various branches of science. The asymmetric MDS is considered as one such extension originated by Young (1975). His augmented distance model is written as

$$d_{ij}^* = \sqrt{\sum_{t=1}^{r} w_{it}(x_{it} - x_{jt})^2}, \quad w_{it} \geq 0, \tag{3}$$

where w_{it} is the weight of object i on dimension t, x_{it} is the coordinate of object i on dimension t.

Since then, dozens of asymmetric MDS methods have been proposed. The major traditional asymmetric MDS methods include Chino (1978, 1990), Chino and Shiraiwa (1993), Constantine and Gower (1978), Escoufier and Grorud (1980), Gower (1977), Harshman (1978), Harshman, Green, Wind, and Lundy (1982), Kiers and Takane (1994), Krumhansl (1978), Okada and Imaizumi (1987), Saito (1991), Saito and Takeda (1990), Sato (1988), Tobler (1976-77), Weeks and Bentler (1982), Young (1975), Zielman and Heiser (1993).

For example, the **distance-density model** by Krumhansl (1978) is written as,

$$d_{jk}^* = d_{jk} + \alpha\,\delta(\boldsymbol{x}_j) + \beta\,\delta(\boldsymbol{x}_k). \tag{4}$$

where $\delta(\boldsymbol{x}_j)$ and $\delta(\boldsymbol{x}_k)$ are measures of spatial density in neighborhoods of x and y, and α and β are constants that reflect the relative weight given the above densities.

The following three augumented distance models are quite similar with each other. These are the Weeks-Bentler model, the Okada-Imaizumi model, and the Saito-Takeda model, and are written, respectively, as

$$d_{jk}^* = b\,d_{jk} + c_j - c_k + a, \tag{5}$$

$$d^*_{jk} = d_{jk} - r_j + r_k. \quad (6)$$

and

$$d^*_{jk} = d_{jk} + \theta_j + \phi_k + \gamma. \quad (7)$$

Here, the Okada-Imaizumi model is a special case of their most general model, and is a nonmetric MDS. Saito (1991) generalizes the Saito-Takeda model.

The **GIPSCAL model** (a generalized inner product MDS) originated by Chino (1978, 1990) is written as

$$\boldsymbol{S} = a\,\boldsymbol{X}\boldsymbol{X}^t + b\,\boldsymbol{X}\boldsymbol{L}_q\boldsymbol{X}^t + c\,\mathbf{1}_N\mathbf{1}_N^t + \boldsymbol{E}. \quad (8)$$

Here, the matrix $\boldsymbol{L}_q$ is a special $q \times q$ matrix written as

$$\boldsymbol{L}_q = \{\epsilon_{st}\}, \quad (9)$$

where

$$\epsilon_{st} = \begin{cases} 1, & \text{when the } (\cdots st \cdots) \text{ is the even permutation of } (1\,2 \cdots q)\ , \\ 0, & \text{when the two subscripts } s,\, t \text{ are the same}\ , \\ -1, & \text{when the } (\cdots st \cdots) \text{ is the odd permutation of } (1\,2 \cdots q)\ . \end{cases} \quad (10)$$

The special case of GIPSCAL when the number of dimension is less than or equal to 3, was first proposed by Chino (1978). Especially when the space is two-dimensional, interpretation of the configuration obtained by GIPSCAL is very simple. Table 1 shows an artificial data. In this table, assume that the positive value indicates the positive sentiment and the negative value the negative sentiment. Then, for example, member 4 likes member 1 very much, but member 1 dislikes member 4 very much. Such a relationship is represented as shown in figure 3.

Table 1: An artificial asymmetric data between 6 members

rater\ratee	1	2	3	4	5	6
1	1	1	-1	-1	0	$1/2$
2	-1	1	1	-1	$-\sqrt{2}$	$-1/2$
3	-1	-1	1	1	0	$-1/2$
4	1	-1	-1	1	$\sqrt{2}$	$1/2$
5	$\sqrt{2}$	0	$-\sqrt{2}$	0	1	$\sqrt{2}/2$
6	$1/2$	$1/2$	$-1/2$	$-1/2$	0	$1/2$

As shown in this figure, the skewness between members 1 and 4 is the greatest of all, as the angle between the lines M_0M_1 and M_0M_4 is $\pi/2$. For,

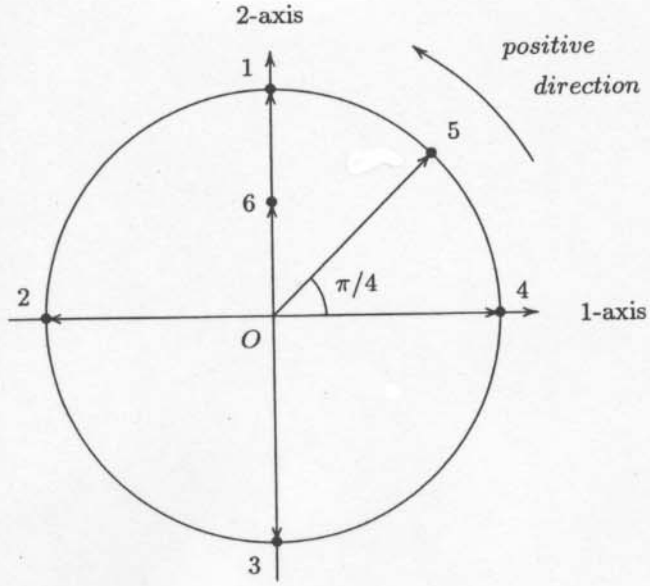

Figure 3: The assumed configuration of 6 members

in GIPSCAL we generally represent the absolute value of the skewness of members i and j by twice the area of the triangle with vertices M_0, e.g., the origin, M_i, and M_j. This idea is the same as that of the Gower Diagram by Gower (1977).

In GIPSCAL, we can read another important information from its graphical output. It is the direction of skewness between members. In this figure, it is counter-clockwise. From this information, it is easy for us to see that member 4 likes member 1, but member 1 dislikes member 4.

The **DEDICOM** (*DE*composition into *DI*rectional *COM*ponents) model proposed by Harshman (1978) is written as

$$\boldsymbol{S} = \boldsymbol{Y}\boldsymbol{A}\boldsymbol{Y}^t + \boldsymbol{E}. \tag{11}$$

where $\boldsymbol{Y}$ is an $n \times p$ loading matrix of objects and $\boldsymbol{A}$ is a $p \times p$ matrix whose elements represent the directional relationships between dimensions.

The **Complex Coding** proposed by Escoufier & Grorud (1980), which I later call the Hermitian canonical model, is written as

$$s_{jk}(s) \approx \lambda_1 \, (u_{j1}u_{k1} + v_{j1}v_{k1}), \tag{12}$$

$$s_{jk}(sk) \approx \lambda_1 \, (v_{j1}u_{k1} - u_{j1}v_{k1}), \tag{13}$$

where $s_{jk}(s)$ and $s_{jk}(sk)$ are are the symmetric part and the skew-symmetric part of the original similarity data matrix $\boldsymbol{S}$, respectively. Note that

$$\boldsymbol{S} = \{s_{jk}\} = \frac{1}{2}(\boldsymbol{S} + \boldsymbol{S}^t) + \frac{1}{2}(\boldsymbol{S} - \boldsymbol{S}^t) = \boldsymbol{S}_s + \boldsymbol{S}_{sk}. \tag{14}$$

Chino & Shiraiwa (1993) deduced essentially the same model independently using the different approach from their approach, and called it HFM (the **Hermitian Form Model**). This is written as

$$\boldsymbol{H} = \boldsymbol{X}\boldsymbol{\Omega}_s\boldsymbol{X}^t + i\,\boldsymbol{X}\boldsymbol{\Omega}_{sk}\boldsymbol{X}^t, \tag{15}$$

where $\boldsymbol{H}$ is a *Hermitian matrix*,

$$\boldsymbol{H} = \boldsymbol{S}_s + i\,\boldsymbol{S}_{sk}, \tag{16}$$

constructed uniquely from any square asymmetric matrix $\boldsymbol{S}$, and

$$\boldsymbol{\Omega}_s = \begin{pmatrix} \boldsymbol{\Lambda}, & \boldsymbol{O}_3 \\ \boldsymbol{O}_3, & \boldsymbol{\Lambda} \end{pmatrix}, \quad \boldsymbol{\Omega}_{sk} = \begin{pmatrix} \boldsymbol{O}_3, & -\boldsymbol{\Lambda} \\ \boldsymbol{\Lambda}, & \boldsymbol{O}_3 \end{pmatrix}. \tag{17}$$

Equation (15) is rewritten, in real form, as

$$s_{ij} = \sum_{l=1}^{r} \lambda_l\,(r_{il}r_{jl} + c_{il}c_{jl}) + \sum_{l=1}^{r} \lambda_l\,(c_{il}r_{jl} - r_{il}c_{jl}), \tag{18}$$

Chino and Shiraiwa (1993) have shown that the complex counterparts of DEDICOM, GIPSCAL, HCM (e.g., Complex Coding), and other possible models of $\boldsymbol{H}$ are expressible in terms of *finite-dimensional complex Hilbert space* if $\boldsymbol{H}$ is p.s.d. This result motivates the **Chino-Shiraiwa theorem**, which is a generalization of the Young-Householder theorem to the complex case.

According to this theorem, a necessary and sufficient condition for a set of numbers d_{ij} $(= d_{ji})$ and $\bar{d}_{ij}$ to be the mutual distances of a real set of points in *Hilbert space* is that the *Hermitian scalar product matrix* $\boldsymbol{H}$ whose elements h_{ij} are define by

$$h_{ij} = \frac{1}{2}(d_{io}^2 + d_{jo}^2 - d_{ij}^2) + \frac{1}{2}\,i\,(d_{io}^2 + d_{jo}^2 - \bar{d}_{ij}^2), \quad 1 \le i,\, j \le n. \tag{19}$$

be p.s.d., and in this case the set of points is unique apart from a *Unitary transformation*. Here, the subscript o indicates the origin.

Here, d_{ij} and $\bar{d}_{ij}$ are defined, respectively, as

$$d_{ij} = \|\boldsymbol{v}_i - \boldsymbol{v}_j\|, \quad 1 \le i,\, j \le n, \tag{20}$$

and

$$\bar{d}_{ij} = \|\boldsymbol{v}_i - i\,\boldsymbol{v}_j\|, \quad 1 \le i,\, j \le n, \tag{21}$$

where $\boldsymbol{v}_i (1 \ge i \ge n)$ be the row vector of order r, which corresponds to the i-th row of $\boldsymbol{U}_1$ in the following equation

$$\boldsymbol{H} = \boldsymbol{U}_1\boldsymbol{\Lambda}\boldsymbol{U}_1^*. \tag{22}$$

Here, $\boldsymbol{U}_1$ is a semi-unitary matrix composed of the r column vectors corresponding to the eigenvectors associated with the non-zero eigenvalues of $\boldsymbol{H}$. $\boldsymbol{U}_1^*$ is the conjugate transpose of $\boldsymbol{U}_1$. Moreover, $\boldsymbol{\Lambda}$ is the diagonal matrix whose diagonal elements are these eigenvalues with the descending order. It should be noticed that the following two equations hold for equations (20) and (21):

$$d_{ij} = d_{ji}, \quad 1 \leq i, j \leq n, \quad d_{ii} = 0, \quad 1 \leq i \leq n, \tag{23}$$

$$d_{io}^2 + d_{jo}^2 - \overline{d}_{ij}^2 = -(d_{jo}^2 + d_{io}^2 - \overline{d}_{ji}^2), \quad 1 \leq i, j \leq n, \tag{24}$$

As far as we are concerned with the spatial representation of the asymmetric relationships contained in the data, it is unnecessary to consider the complex space structure shown in this theorem, and thus we may apply the Complex Coding to the data. However, if we examine the dynamical aspect of the data, that is, changes in the distance structure over time, it will be very useful and convenient to utilize such a complex structure. For, we may utilize the theories of the **complex difference system** in mathematics. Later, I will show you a preliminary application of this system to the interpersonal attraction data.

Recently, some researchers have been extending the asymmetric MDS further. For example, Okada and Imaizumi (1997) extended their two-way nonmetric MDS to a three-way case. Yadohisa and Niki (1999) propose a vector field representation of asymmetric proximity data, especially the scalar potential of the field. Trendafilov (2002) reformulates the GIPSCAL into an initial value problem for matrix ordinary differential equations on manifolds defined by the constraints of the original least-squares problems. Rocci and Bove (2002) generalize the Complex Coding, and discuss relations to other works of Chino (1990), Harshman (1978), and Kiers and Takane (1994).

Anyway, I will not go into details of the recent developments in the asymmetric MDS for the time limitation. Instead, I will concentrate on some possible developments in the future on the asymmetric MDS, which I have partly been developing with my colleagues. One of them is the developments of some inferential methods and statistics for the asymmetric MDS. Another is concerned with the linkage of the asymmetric MDS to the analysis of contingency tables by developing further the successive categories symmetric multidimensional scaling originated by Takane (1981). The other is concerned with the linkage of the asymmetric MDS to the dynammical system theories in mathematics. As for this, I will show you two preliminary studies, one utilizing the qualitative theories of **differential dynamical systems** and the other using the **difference dynamical systems**.

2 Possible developments in the future

Now, I will show you three possible developments in the future.

2.1 Developments of some inferential methods and statistics

As discussed above, dozens of asymmetric MDS models have been proposed since the work of Young. However, the asymmetric MDS has remained a discriptive method, whereas the symmetric MDS has some inferential methods based on the maximum likelihood (ML) method (for example Ramsay, 1977; Takane, 1981). Although Chino (1992) proposed the framework of an ML method for the asymmetric MDS which extended Takane's (1981) method to asymmetric data, its algorithm has not been completed yet. However, Saburi & Chino (2004) have recently developed Chino's (1992) method further. We call it ASYMMAXSCAL.

ASYMMAXSCAL has three major advantages. *First*, it enables us to test the symmetry hypothesis for the given similarity data. *Second*, we can compare the goodness of fit of the data among some extant models for the asymmetric MDS using the Akaike information criterion, AIC (Akaike, 1973, 1977). *Third*, it allows us to employ not only the metric data but also non-metric data. We shall summarize it next.

In this method, we consider the three models, i.e., the *representation model*, *the error model*, and the *response model*. Any model proposed previously for the asymmetric MDS may be basically chosen as the representation model. In this study, we choose HFM discussed above. So, we shall call it ASYMMAXSCAL-HFM. In order to distinguish the s_{ij} as the error perturbed data with those as the true value, we shall revise the **representation model** as

$$g_{ij} = \sum_{l=1}^{r} \lambda_l \left(r_{il} r_{jl} + c_{il} c_{jl} \right) + \sum_{l=1}^{r} \lambda_l \left(c_{il} r_{jl} - r_{il} c_{jl} \right), \tag{25}$$

indtead of equation (18) in this context.

In this way, the similarity g_{ij} in our model is assumed to be error-perturbed by some psychological process. In this method g_{ij} can be either positive or negative theoretically, because it is a kind of inner product. Thus, we consider only the additive error model of the two **error model**s proposed by Takane (1981), i.e.,

$$\tau_{ijk}^{(t)} = g_{ij} + e_{ijk}^{(t)}, \qquad e_{ijk}^{(t)} \sim N\Big(0, \sigma_k^2\Big), \tag{26}$$

where $\tau_{ijk}^{(t)}$ is a psychological value for subject k corresponding to the proximity from object i to object j at replication t.

As a **response model**, we employ the successive categories scaling model (Torgerson, 1958), following Takane (1981). Let a rating scale be composed of M observation categories $C_1, C_2, \ldots, C_M$, and let b_{km} represent the upper boundary of the m-th category for subject k. Under the normal distribution assumption made above, we may assume

$$-\infty = b_{k0} \leq \cdots \leq b_{km} \leq \cdots \leq b_{kM} = \infty, \tag{27}$$

and define the probability of a subject's response by

$$p_{ijkm} = \Pr(o_{ijkr} \in C_m) = \Pr(b_{k(m-1)} < \tau_{ijk} \le b_{km}) = \int_{b_{k(m-1)}}^{b_{km}} f(\tau_{ijk}) d\tau_{ijk}, \tag{28}$$

where o_{ijkr} is the proximity from object i to object j for subject k at replication r, and f is the density function of the normal distribution.

Here we make three assumptions concerning these category boundaries, accoding to Takane (1981): linear constraints without individual differences, unrestricted with individual differences, and completely unrestricted. The latter two contraints correspond to a nonmetric case.

In order to estimate the vector of all parameters $\boldsymbol{\theta}$, we may maximize the logarithm of the joint likelihood of the observations,

$$\log L = \sum_k \sum_{i,\ j} \sum_m Y_{ijkm} \log p_{ijkm}, \tag{29}$$

where $Y_{ijkm} = \sum_r Z_{ijkmr}$, and Z_{ijkmr} is the subject's response coded as

$$Z_{ijkmr} = \begin{cases} 1, & when\ o_{ijkr} \in C_m, \\ 0, & otherwise. \end{cases} \tag{30}$$

The maximization procedure is essentially the same as that of Takane (1981). That is, we estimate the unrestricted parameters $\boldsymbol{\theta}^*$ by Fisher's scoring method.

Then, we may test a **symmetry hypothesis** $H_0^{(1)}$:

$$c_{ia} = 0, \quad i = 1, 2, \ldots, n;\ a = 1, 2, \ldots, A, \tag{31}$$

where n is the number of the objects. This hypothesis is equivalent to saying that there are no asymmetric components in the data. We use a well-known χ^2 statistic. This test completely depends on the three models, that is, the representation model, the error model, and the response model.

We may construct another symmetry hypothesis. This is accomplished by introducing another representation model. We call it a **saturated representation model**, in which g_{ij} themselves are regarded as the parameters to be estimated from the data. In this case, the test depends only on the error model and the response model. In this case, the symmetry hypothesis is defined as

$$g_{ij} = g_{ji}, \qquad (1 \le i < j \le n). \tag{32}$$

If we choose the test of this hypothesis, we can test it prior to the estimation of all the parameters of the model.

For the given similarity data, we can compare the goodness of fit among some representation models, using AIC defined by

$$\mathrm{A}IC = -2 \log L + 2\mathrm{d}.f., \tag{33}$$

where d.f. is the effective number of parameters in a model (Akaike, 1973, 1977).

Now, we shall show you an example of the application of our method to a set of the data gathered by Saburi and Chino (2005). 163 subjects watched an animation movie called "Kurenai no Buta" which means "The Crimson Pig" produced by Hayao Miyazaki, a famous Japanese animation film producer, in which interpersonal relationships develop gradually between 5 major characters. The leading character is called Porco, who is an airborne pilot on the Adriatic in the 1920's. At the middle and the end points of the movie, subjects rated how characters liked or disliked one another including

Table 2: Cross-classified table for 5 characters in Category 2 (dislike fairly)

rater\ratee	1.Boss	2.Curtis	3.Fio	4.Gina	5.Porco	total
1.Boss	3	26	0	0	40	69
2.Curtis	8	1	1	1	35	46
3.Fio	19	36	0	1	0	56
4.Gina	3	9	2	7	0	21
5.Porko	15	11	0	0	28	54
total	48	83	3	9	103	246

Table 3: Cross-classified table for 5 characters in Category 6 (like fairly)

rater\ratee	1.Boss	2.Curtis	3.Fio	4.Gina	5.Porco	total
1.Boss	37	2	53	46	2	140
2.Curtis	1	44	61	67	0	173
3.Fio	1	1	46	14	60	122
4.Gina	0	1	7	15	53	76
5.Porko	4	1	75	56	12	148
total	48	83	3	9	103	659

themselves on 7-point rating scales. Tables 2 and 3 show two-way frequency tables for categories C_2 (dislike a person fairly) and C_6 (like a person fairly), respectively.

We first chose the saturated model as a representation model and computed its estimates. Then we tested the symmetry hypothesis and found it to be rejected under both the ordinal and interval scale assumptions ($\chi^2(10) = 1966{\cdot}76$, $p < 0{\cdot}0001$; $\chi^2(10) = 1842{\cdot}89$, $p < 0{\cdot}0001$, respectively). Next, we chose HFM as another representation model and computed its estimates under the unitary constraint. The minimum value of AIC was attained in the case of three dimensionality under the ordinal scale assumption. In HFM, one (complex) dimension is equivalent to two-dimensions in the real space. Figure 4 is the estimated configuration of 5 characters with the 95% asymptotic confidence regions for each of the locations of objects in the first complex dimension.

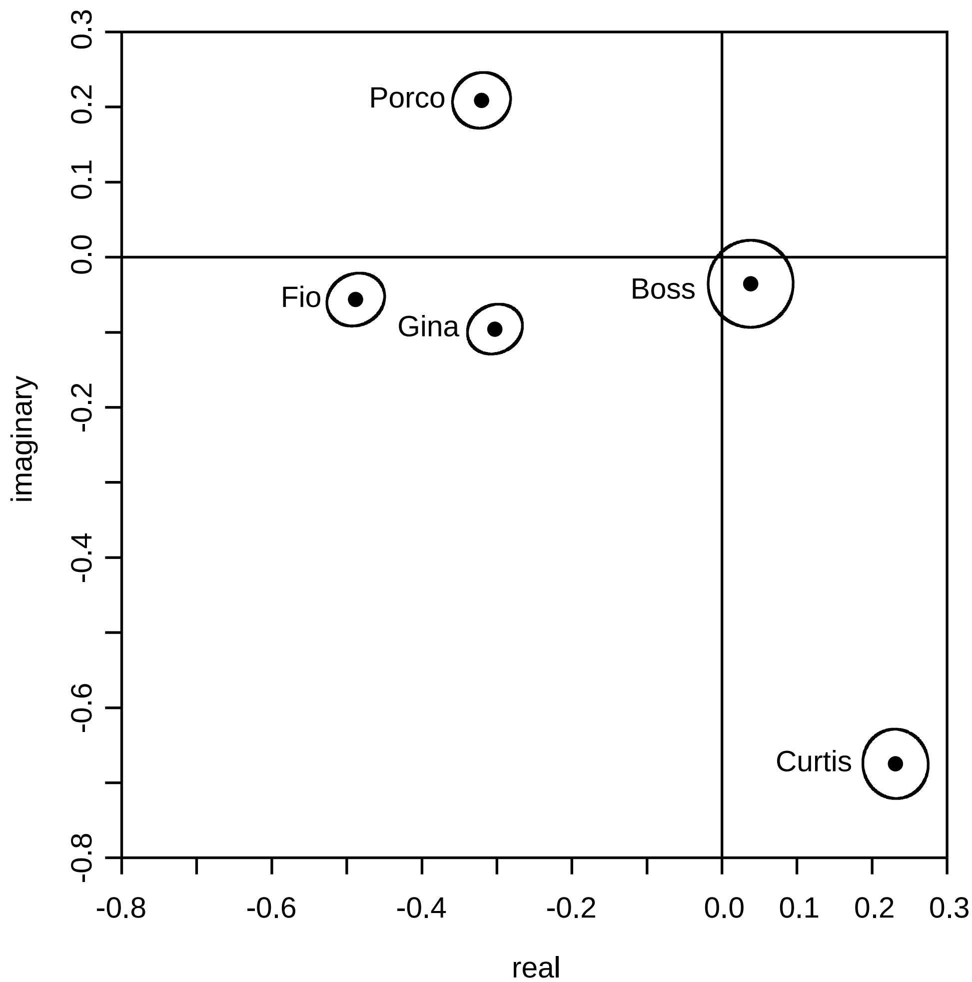

Figure 4: Stmulus configuration of the first dimension with 95% confidence regions

In interpreting the configuration obtained by HFM, it should be noted that the direction of the skew-symmetry can be determined by the sign of the weight of the model as is the case with GIPSCAL discussed above, and that the positive direction is clockwise if the weight is positive in contrast with GIPSCAL. Since the estimated weight was positive for the first dimension, this direction is clockwise in this case. From figure 4, we see, for example, that Curtis likes Fio and Gina very much, and conversely they dislike him very much.

2.2 Linkage to the analysis of contingency tables

Another possible development of the asymmetric MDS may be to consider the linkage of the asymmetric MDS to the analysis of contingency tables. As discussed above, our symmetry hypotheses are in a sense restrictive in that these hypotheses are dependent on some or all of the models in ASYMMAXSCAL. This is partly because we don't restrict the data within count data.

However, introduction of the idea of the successive categories scaling in MDS enables us to build the bridge between qualitative data, e.g, the count data, and the quantitative data, e.g., data with the ordinal, intervarl, or ratio level of measurement. In fact, as I have presented at the talk of the 2nd German-Japanese Symposium on Classification the other day, the data which our ASYMMAXSCAL requires are considered as the result of the Type A design of Figure 5.

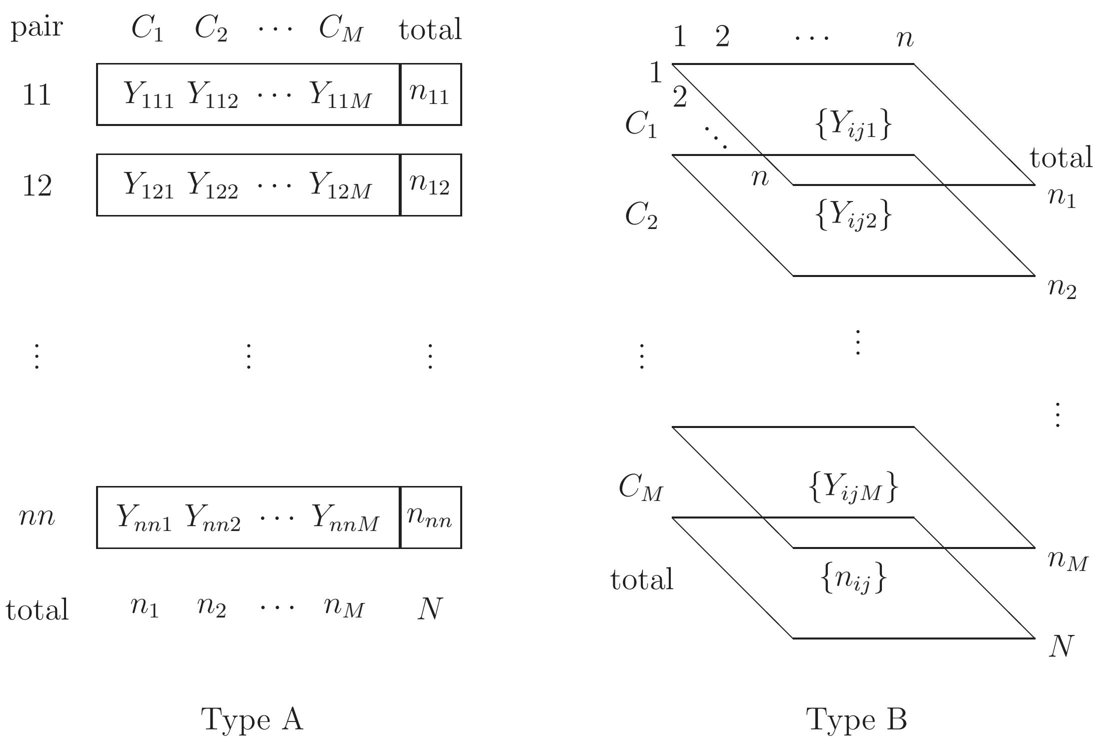

Figure 5: Two types of the sampling design for the method of successive categories scaling in ASYMMAXSCAL

The Type A design assumes that each row $(Y_{ij1}, Y_{ij2}, \ldots, Y_{ijM})$ follow, say, a multinomial distribution with parameter n_{ij} which is fixed and certain cell probabilities $p_{ij1}, p_{ij2}, \ldots, p_{ijM}$. This design is nothing but the sampling design peculiar to our ASYMMAXSCAL as well as MAXSCAL by Takane (1981).

By contrast, the Type B design assumes that all the entries of each of the two-way contingency tables, that is, $\{Y_{ijm}\}$ follow, say, a multinomial distribution with parameter $n_m = \sum_i^n \sum_j^n Y_{ijm}$ which is fixed and certain cell probabilities $p_{11m}, p_{12m}, \ldots, p_{nnm}$, where m indicates the mth rating category. Moreover, it is assumed that M two-way contingency tables are mutually independent. This design is the traditional design for the special three-way contingency table, that is, the $n \times n \times M$ table (Bishop, Fienberg, & Holland, 1975; Read, 1978).

These two designs differ somewhat, and generally lead to slightly different LR χ^2 statistics. Recently, Saburi & Chino (2006, submitted paper) have proven a relationship between the two statistics.

Anyway, we can test a third symmetry hypothesis, e.g., **conditional symmetry hypothesis**,

$$H_0^{(cs)}:\ p_{ijm} = p_{jim}, \qquad (1 \leq i < j \leq n;\ 1 \leq m \leq M-1). \tag{34}$$

Although Read calls the corresponding hypothesis "symmetry within levels", we shall follow Bishop, Fienberg, and Holland (1975) and call it the conditional symmetry. This kind of conditional symmetry test was suggested by Takane (2005, personal communication). This hypothesis is completely independent on any of the threee models of ASYMMAXSCAL.

Given a *two-way square contingency table*, Zielman and Heiser (1996) suggest a usual test of symmetry originated by Bowker (1948) in discussing the models of asymmetric proximities, while de Rooij & Heiser (2003, 2005) test not only this symmetry hypothesis but also a usual quasisymmetry hypothesis. By contrast, we assume a more general situation where the data is obtained either at an ordinal, interval, or ratio level. Moreover, our target of the analysis is on the *special three-way table*.

Of course, we can utilize the traditional key concepts on the analysis of contingency table in statistics further. In this case, such an analysis can be viewed as a preprocessing step of our ML asymmetric MDS, that is, ASYMMAXSCAL. The test of conditional symmetry suggested by Takane is one such analysis. If we analyze this table in line with this, we can utilize some basic, and important tests for symmetry and related tests in addition to the symmetry test suggested by him.

These tests are those of **quasisymmetry**, **marginal homogeneity**, **quasi-independence**, **independence**, and a **log-linear hypothesis**. Combinations of these tests enable us not only to diagnose the validity of applying any asymmetric MDS to the asymmetric relational data matrices but also to examine various types of symmetry and asymmetry contained in square contingency tables. Such a result might serve as an indirect examination of the cause of asymmetry contained in the $n \times n \times M$ contingency table. The notion of quasisymmetry introduced by Caussinus (1965) plays an important role in such an examination, as shown in Figure 6. <Fig.6

Figure 6 illustrates a hierarchical structure of these tests. As pointed out, for example, in Andersen (1980), there exists a strict **hierarchical order** in some of these tests as to which hypotheses must be tested first and which hypotheses must be tested under the assumption that other hypotheses hold. This order is, of course, concerned with the tests of the log-linear hypotheses on independence, that is, H_I, H_1, H_2, in Figure 6.

Considering the purpose of applying some asymmetric MDS to the relational data, it will be necessary and appropriate to test the conditional symmetry discussed above, first. If it is accepted, we had better apply a certain traditional **symmetric MDS** to the data. If the conditional symmetry hypothesis stated above is rejected, we may proceed to the test for quasisymmetry. As we are now dealing with a three-way table, it may be appropriate to call it the test for **conditional quasisymmetry**. Then, we may proceed to some appropriate tests following the flow shown in Figure 6, starting from this hypothesis.

In this figure, various hypotheses are indicated in the lozenge diagram. H_{QS} means the quasisymmetry hypothesis, H_{QI} the quasiindependence, H_{MH} the marginal homogeneity, H_I the usual independence, H_1 the row effect and H_2 the column effect of the log-linear model.

At a glance, we can notice the followings:

1. There exist four types of symmetry under the quasisymmetry and the marginal homogeneity hypotheses:

   a) R_1 – independent, neither the row nor the column effect exists.

   b) R_4 – independent, both the row and the column effects exist.

   c) R_6 – dependent.

   d) R_8 – quasidependent.

2. There exist five types of asymmetry when the quasisymmetry hold but the marginal homogeneity does not hold:

   a) R_2 – independent, and the column effect only.

   b) R_3 – independent, and the row effect only.

   c) R_5 – independent, and both of the row and the column effects hold.

   d) R_7 – dependent.

   e) R_9 – quasidependent.

3. There exist the other two types of asymmetry when the quasisymmetry does not hold:

   a) R_{10} – the marginal homogeneity holds.

   b) R_{11} – the marginal homogeneity does not hold.

It is interesting to note that the causes of statistical asymmetry under the quasisymmetry hypothesis are restricted within the **row** and the **column** effects or **dependency** of ratings between rater and ratee. Moreover, it is important to notice that if the conditional quasisymmetry hypothesis is accepted, it will be appropriate to choose one of the augmented distance models for asymmetric MDS as a subsequent analysis because they are congruous with the quasisymmetry hypothesis. Otherwise, it will be necessary to choose one of the non-augmented distance models such as GIPSCAL, Complex coding or HFM, Sato's asymmetric Minkofsky metric model.

Now, we shall show an example of the application of the hierarchical tests discussed above to a set of the data gathered by Saburi and Chino (2005), two of which have already been shown in this talk. These are examples of the two-way tables shown as the Type B data in Figure 5. Strictly speaking, however, such a sampling might not necessarily fulfill the assumption of mutual independences of the judgments for all pairs of objects in principle.

Therefore, it might be necessary to check whether these judgments are independent or not. To do so, we performed the usual Pearson type χ^2 test of independence of the ratings, shown as Type A design data in Figure 5, for all the combinations of pairs of objects. Since the number of objects is five, we have 20 combinations of pairs, and 190 7×7 contingency tables. Since many of these contingency tables included random zeros, we used the Yates' correction for continuity. As a result, about 85 percentages of these contingency tables were considered to be independent.

Table 4: Tests of symmetry for the Crimzon Pig data

χ^2, p-value, & d.f.	G^2	p-value	X^2	p-value	d.f.
C_1	55.69	p<0.001	45.26	p<0.001	10
C_2	115.29	p<0.001	95.26	p<0.001	10
C_3	175.58	p<0.001	137.57	p<0.001	10
C_4	329.68	p<0.001	289.84	p<0.001	10
C_5	203.99	p<0.001	180.93	p<0.001	10
C_6	294.76	p<0.001	224.28	p<0.001	10
C_7	274.66	p<0.001	216.45	p<0.001	10
total	1449.65	p<0.005	1189.59	p<0.005	70
(revised)					(60)

Tables 4 and 5 are the results of the tests of symmetry and quasisymmetry, respectively. Not only the overall symmetry hypothesis, that is, the conditional symmetry hypothesis, but also the component symmetry hypotheses are rejected as shown in Table 4. Therefore, we may proceed to the test of quasisymmetry. Table 5 shows the result. The overall test of quasisymmetry, that is, the conditional quasisymmetry, is rejected, even if we administer the Yates' correction of continuity to the data. In such a case, we generally had better choose some non-augmented distance model. However, Table 5 shows a somewhat complicated result in that some of the component quasisymmetry hypotheses hold for this set of data.

Table 5: Tests of quasisymmetry for the Crimzon Pig data

χ^2, p-value, & d.f.	G^2	p-value	X_Y^2	p-value	d.f.
C_1	5.42	-	0.58	-	6
C_2	8.76	-	4.15	-	6
C_3	9.07	-	5.03	-	6
C_4	68.92	p<0.001	50.20	p<0.001	6
C_5	36.30	p<0.001	31.99	p<0.001	6
C_6	32.63	p<0.001	46.66	p<0.001	6
C_7	0.49	-	0.00	-	6
total	161.60	p<0.005	138.61	p<0.005	42
(revised)					(36)

X_Y^2 indicates the Pearson χ^2 with Yates' correction for continuity.

Table 6: LR $\chi^2(G^2)$ for various symmetry and a related test of the cross tables for C_2 & C_6 by SAS, which are shown in Tables, 1 and 2

hypothesis	G^2 for C_2	G^2 for C_6	d.f.
1.H_s	115.29, p<0.001	294.76, p<0.001	10
2.H_{QS}	8.76	32.63, p<0.001	6
3.H_{QI}	90.23, p<0.001	186.82, p<0.001	11
4.H_I	176.43, p<0.001	0.00, p<0.001	16
5.H_1	29.28, p<0.001	704.69, p<0.001	4
6.H_2	189.28, p<0.001	452.14, p<0.001	4
7.H_{MH}	293.747, p<0.001	(659.00), p<0.001	3

In this table, (659.00) indicates that the LR χ^2 statistic cannot be estimated due to error, so the Pearson χ^2 was computed in place of it.

Therefore, we have tentatively tested some other hypotheses for each of the two-way contingency tables corresponding to each of the rating categories, according to the flow chart drawn in Figure 6. Table 6 shows the LR $\chi^2(G^2)$ for various symmetry tests and a related test for cross tables for rating categories C_2 and C_6.

As for category C_2, that is, judgments of "dislike a person fairly", the quasisymmetry hypothesis holds. Thus, we proceed downward to test the subsequent hypotheses according to Figure 6. This examination leads us to the conclusion that the component quasisymmetry hypothesis holds for this table, but the marginal homogeneity does not.

As regards category C_6, that is, judgments of "like a person fairly", none of the hypotheses under study holds, as shown in the right-hand side of Table 6. That is, neither the quasisymmetry nor the marginal homogeneity holds for this table.

2.3 Linkage to the dynamical system theories

The other possible development of the asymmetric MDS may be to consider the linkage of the asymmetric MDS to the dynamical system theories in mathematics.

2.3.1 Differential dynamical system

I will first talk about a preliminary work conducted by Chino & Nakagawa about twenty years ago (Chino & Nakagawa, 1983, 1990). We call it the DYNASCAL, which is an abbreviation of the DYNAmical system SCALing. <Tab.7

Before proceeding, I wiil show you an example of the data for this method. About fourty years ago, famous social psychologist, T. M. Newcomb observed an acquaintance process of 17 previously unacquainted male students, who lived together in a fraternity-style house, expenses paid. The obtained self-reports of complete rank orderings of the other 16 by attraction during each of the 16 weeks of experiment are a set of longitudinal sociomatrices. Table 7 shows the sociomatrices at Weeks 1 and 2.

If we apply a certain asymmetric MDS to each of these sociomatrices, we obtain a set of longitudinal configurations of members. A glance at these configurations will raise various questions concerning the group formation processes:

1. What is the fundamental law which governs the formation processes?
2. How can we uncover the underlying dynamics of the system composed of members who interact with each other?
3. What kinds of dynamics are theoretically possible in such a system?
4. What kinds of dynamics are observable and what kinds of dynamics are not in such a system?
5. What are the determinants of group formation and dissolution?
6. Is it possible to predict and control group formation-dissolution processes?

DYNASCAL answers some of such questions by estimating the latent qualitative as well as quantitative aspects of the dynamical system, given a set of longitudinal asymmetric relational data matrices. To do this job, DYNASCAL assumes the followings:

1. **Dimension of the space**
   For simplicity, we assume that the dimension of the space is 2.

2. **State variables**
   State variables of our system are coordinates of objects at each time, and are estimated from data.

3. **Hyperbolicity**
   This assumption guarantees that our system is *locally structurally stable*, and may be rational since orbital property of a structurally unstable system changes by a small perturbation and therefore can not be *observed* by any method which accompanies *measurement errors* or *computation errors*.

4. **Boundary condition** — closed region whose boundary is a simple smooth curve
   This is a necessary condition for the system to be *globally structurally stable* under some general conditions (Peixoto, M. C., and Peixoto, M. M., 1959).

5. **System** — general nonlinear nonautonomous system:

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} f(x,y,t) \\ g(x,y,t) \end{pmatrix}. \tag{35}$$

   Here, we assume that m bifurcation parameters of the *following autonomous system* have all been projected onto the time axis, t, of the above system. Thus, the *above nonautonomous system* will be revised at any time when we will be able to identify such parameters. Moreover, we will neither specify the forms of f nor g *a priori*. Instead, we will *estimate these forms indirectly from data*, that is, a set of longitudinal relational data matrices:

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} f(x,y,c_1,c_2,\cdots,c_m) \\ g(x,y,c_1,c_2,\cdots,c_m) \end{pmatrix}. \tag{36}$$

Now, at any instant of time, $t = c$, the specified system, that is, Eq. (35), can be viewed as the following nonlinear *autonomous* system:

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}. \tag{37}$$

Then, its local orbital properties can be examined by **linearizing the system** at a special point $\boldsymbol{x}^*$ called **singularity** or **singular point** defined as follows (Hirsch & Smale, 1974):

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \boldsymbol{J}(\boldsymbol{x}^*) \begin{pmatrix} x \\ y \end{pmatrix}, \tag{38}$$

where

$$\boldsymbol{J}(\boldsymbol{x}^*) = \begin{pmatrix} \partial f/\partial x, & \partial f/\partial y \\ \partial g/\partial x, & \partial g/\partial y \end{pmatrix}_{\boldsymbol{x}=\boldsymbol{x}^*}. \tag{39}$$

$\boldsymbol{J}(\boldsymbol{x}^*)$ is called the Jacobian of singularity $\boldsymbol{x}^*$.

Here, in general, a singularity is the point at which first-derivatives all vanish. In the case of Eq. (37), the $\boldsymbol{x}^*$ is a solution of

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{40}$$

If the determinant of the Jacobian is zero at a singularity, it is called the **degenerate singularity**. <Fig. 7

Figure 7 shows fundamental nondegenerate singularities and qualitative behaviors of the system near these singularities. Here, it should be noted that one of them, e.g., the center, cann't be observed in principle. The reason is as follows:

According to the Hartman and Grobman theorem (for example, Guckenheimer & Holmes, 1983), the vector field is **locally structurally stable** if the Jacobian of the system at a singularity has non-zero real part (that is, if the singularity is hyperbolic). In figure 7, only the "center" has zero real parts. Thus, the "center" is structurally unstable and **can't be observed**. Degenerate singularities are also structurally unstable. <Fig. 8

Nonlinear systems like Eq. (37) have another sort of local orbital property. It is called the **limit cycle**. Figure 8 shows two fundamental limit cycles. We can also interpret these behaviors of limit cycles psychologically (Chino, 1988).

Let us now suppose that there exists a certain factor which influences the force field under study and the value of the factor changes continuously by the complicated interactions among individuals or some influence from outside the system. Mathematically, such a factor is called a bifurcation parameter. Temperature μ is one such example in Hopf bifurcation.

Here, it should be remembered that our system described by Eq. (35) is nonautonomous, and thus it has no bifurcation parameter. However, one can think of successive vector fields described by the nonautonomous system *as changes in the vector field described by an autonomous system over time*. Then, no problem arises in applying the bifurcation theory to the estimated system. Thus, let us for a while consider the following autonomous differential equation with one bifurcation parameter, c:

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} f(x,y,c) \\ g(x,y,c) \end{pmatrix}. \tag{41}$$

Four kinds of bifurcations are the best known local bifurcations which depend on a single parameter in a two-dimensional system (Guckenheimer et al., 1983). Two of them is shown in Figure 9. Another type of bifurcation <Fig. 9

<Fig. 10

is called the global bifuracation. They are shown in Figure 10.

Finally, I will show you an example of application of DYNASCAL to the Newcomb data. As a preprocessing of original ranking data, MULTISCAL (Ramsay, 1982) was applied to each of the 15 sociomatrices (Week 9 was missing in Newcomb's data) in this case, and longitudinal two-dimensional configurations were obtained. <Fig. 11

Application of DYNASCAL yields 15 estimated vector fields corresponding to 15 weeks (Chino, 1984). However, I will show only two of them. Figure 11 show these vector fields at Weeks 0 and 15, respectively. Traditional sociograms corresponding to them are also shown in these figures (Chino, 1984). In these vector fields, numbers indicate the locations of members, and vectors show the estimated velocity vectors at some grid points. Some orbits of the estimated system are shown to facilitate understanding of the global qualitative features of each of the vector fields.

In Figure 11-above, we have two singularities, a **saddle** and a **source**. The saddle clearly divides the left-upper part and the right-lower part of the space. The former includes members 1, 5, 6, 8, 13. The latter includes almost all remaining members except member 10. These results almost coincide with those obtained by depicting traditional sociograms for this set of data.

An important point is that estimated vector fields in general *convey much more dynamic information* about each individuals as well as group structure than traditional sociograms. For example, the vector field in Figure 11 tells us that member 10 is under an unstable circumstance in the sense that only a small change in location leads him either to the left upper part or to the right lower part of the psychological space. This reflects the **instability** of the singularity, saddle.

Another dynamic information is obtained by noting an *omega* **limit cycle** in the upper region of the field. This limit cycle dissapears at the next week. Therefore, we may suppose that the once born group formation process is dissolving. By contrast, Figure 11-below may indicate that the overall group dissolution processes are going on.

2.3.2 Difference dynamical system

If we assume that the state space in which members interact is complex, one way is to utilize some difference equation models instead of differential equation models. I will summarize a preliminary work of mine which is concerned with this kind of models (Chino, 2005).

As discussed above, DYNASCAL has several advantages over some traditional methods for analyzing group structures, i.e., sociograms and Markov process models for social networks (for example, Holland & Leinhardt, 1977). On the other hand, DYNASCAL has several disadvantages, too.

Firstly, it presupposes *asymmetric relationships* between members but the estimated relationships are symmetric. Secondly, it might not be fully justified mathematically to administer the Procrustes rotations to the neighboring pairs of configurations. Thirdly, DYNASCAL will not capture the so-called *chaotic behaviors* since it is restricted to a two-dimensional differential system. Fourthly, it is not possible for DYNASCAL to examine the behaviors of the system theoretically, since it merely estimates the solution curves using spline functions.

To overcome these difficulties, Chino (2002, 2003a) has proposed some **complex diference system models** for social interaction. The most general form of these models is a **general nonlinear model** written as

$$\begin{aligned} \boldsymbol{z}_{j,n+1} &= \boldsymbol{z}_{j,n} \\ &\quad + \sum_{m=1}^{r} \sum_{k \neq j}^{N} \boldsymbol{D}_{jk,n}^{(m)} \boldsymbol{f}^{(m)}(\boldsymbol{z}_{k,n} - \boldsymbol{z}_{j,n}), \\ &\qquad j = 1, 2, \cdots, N, \end{aligned} \tag{42}$$

where,

$$\boldsymbol{f}^{(m)}(\boldsymbol{z}_{k,n} - \boldsymbol{z}_{j,n}) = \begin{pmatrix} \left(z_{k,n}^{(1)} - z_{j,n}^{(1)}\right)^m \\ \left(z_{k,n}^{(2)} - z_{j,n}^{(2)}\right)^m \\ \vdots \\ \left(z_{k,n}^{(p)} - z_{j,n}^{(p)}\right)^m \end{pmatrix}. \tag{43}$$

Moreover, $\boldsymbol{D}_{jk,n}^{(m)} = diag\left\{w_{jk,n}^{(1,m)}, \cdots, w_{jk,n}^{(p,m)}\right\}$, and

$$\begin{aligned} w_{jk,n}^{(l,m)} &= a_n^{(l,m)} r_{j,n}^{(l,m)} r_{k,n}^{(l,m)} \sin(\theta_{k,n}^{(l,m)} - \theta_{j,n}^{(l,m)}), \\ &\quad l = 1, 2, \cdots, p, \quad m = 1, 2, \cdots, r. \end{aligned} \tag{44}$$

It should be noticed that the state space of this model is not **real** but **complex**. Moreover, this model is composed of a set of **multivariate complex difference equations**. Our multivariate complex system models for social interaction may be naturally introduced applying the idea of HFM, which is a one-mode two-way asymmetric MDS (Chino & Shiraiwa, 1993), to longitudinal asymmetric relational data.

It is well known that difference equation models sometimes exhibit complicated chaotic behaviors even in the case of a simple **real** nonlinear equation such as, $x_{n+1} = (1 + r)x_n - rx_n^2$, which is the famous **Verhulst process** (for example, Peitgen & Richter, 1986). It is also well known that even a simple *one-dimensional complex* difference system like Mandelbrot's difference equation, $z_{n+1} = z_n^2 + c$, can describe a variety of curious chaotic behaviors. Therefore, it is expected that our model can predict a variety of behaviors among members of a small group if it is appliable to real life situations. Chino (2003b) proposes some preliminary algorithms to fit a special case of this generalized model to the longitudinal relational data matrices.

Compared with data analytic models like DYNASCAL, theoretical models such as the Verhulst process and our difference model permit to examine their theoretical behaviors precisely. In fact, there has already been proposed a similar model of social systems, although it is restricted to two-person systems. Gregersen and Sailer (1993) examine a **metamodel** of two-person social systems described by the following **real** two-dimensional nonlinear difference equation,

$$
\begin{aligned}
x_{n+1} &= r_x^1 x_n^2 + r_y^1 y_n^2 + r_{xy}^1 x_n y_n - u_x, \\
y_{n+1} &= r_x^2 x_n^2 + r_y^2 y_n^2 + r_{xy}^2 x_n y_n - u_y,
\end{aligned} \tag{45}
$$

and find curious chaotic behaviors. It is apparent that these equations include **Mandelbrot's Set** when $r_x^1 = 1$, $r_y^1 = -1$, $r_{xy}^2 = 2$, with the other rs equal to zero, as they note.

Next, we shall discuss some special cases of our general complex difference system model. Consider first a special case of our model described by eq. (42) through (44) when $p = 1$, $m = 2$, and $N = 2$. This is clearly a special two-person system. In this case, eq. (42) can be written as

$$
z_{j,n+1} = a z_{jn}^2 + b z_{jn} + c, \tag{46}
$$

where

$$
\begin{aligned}
a &= w_{jk,n}^{(2)}, \quad b = 1 - w_{jk,n}^{(1)} - 2 w_{jk,n}^{(2)} z_{kn}, \\
& c = w_{jk,n}^{(1)} z_{kn} + w_{jk,n}^{(2)} z_{kn}^2.
\end{aligned} \tag{47}
$$

Now we shall make a strong assumption that the member j completely ignores the relationship with other member k. In other words, we shall assume that a, b, and c defined by equation (47) are all constants. If one notices that our model is a complex space model, it is evident that equation (46) is equivalent to the Mandelbrot's system.

In a somewhat more general case, when $p = 1$, $m = 2$ in the N-person system, eq. (42) can be written as the same as eq. (46), but

$$
a = \sum_{k \neq j}^{N} w_{jk,n}^{(2)}, \quad b = 1 - \sum_{k \neq j}^{N} w_{jk,n}^{(1)} - 2 \sum_{k \neq j}^{N} w_{jk,n}^{(2)} z_{kn}, \tag{48}
$$

and

$$
c = \sum_{k \neq j}^{N} \left\{ w_{jk,n}^{(1)} z_{kn} + w_{jk,n}^{(2)} z_{kn}^2 \right\}. \tag{49}
$$

In fact, the Julia set becomes a unit circle when $a = 1, b = 0$, and $c = 0$ in equation 46 and equation 49. In this case, there are two fixed points, $z = 0,\ 1$ on the disc. It is easily shown that the former is superattracting, while the latter repelling. Figure 12 shows an orbit, which starts from an interior point of the circle (that is, from a point of the Fatou set of the system under consideration), which approaches to the origin in a few iterations.

By contrast, if the orbit starts from the point $z_0 = cos(\pi/21) + i\, sin(\pi/21)$, it rotates on the unit circle, as shown in Figure 13. Theoretically, this orbit

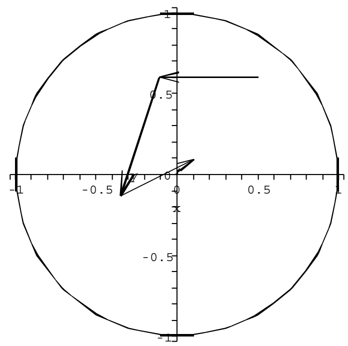

Figure 12: An orbit of the simplest Julia set with the initial value, $z_0 = 0.5 + 0.6i$

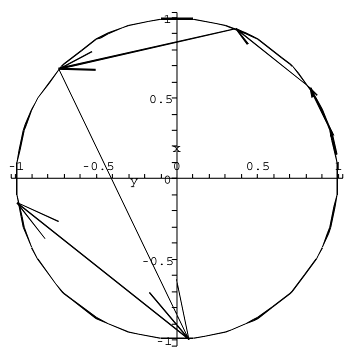

Figure 13: An orbit of the simplest Julia set with the initial value, $z_0 = cos(\pi/21) + i\, sin(\pi/21)$

rotates on the unit circle infinitely, but due to the round errors in computation, it drops in the interior of the circle after a few iterations, and attracted to the origin afterward, as can be seen in Figure 14. Figures 15 and 16, respectively, shows the changes in real- and imaginary-coordinates of the orbit depicted in Figure 14.

As another Mandelbrot process, we show a process with $c = -0.12 + 0.74i$ in Figure 17, which is a well-known process. Its rough picture is, for example, shown in Figure 4 in Peitgen and Richter (1986). The white part of the figure is, of course, the filled Julia set generated by the process, its boundary is the Julia set, and the complement of the Julia set is the Fatou set. In this system, there are two repelling points, $-1.2737+0.4782i$ and $1.2737-0.4782i$. Figure 18 shows an orbit which starts from a neighborhood of the former repelling point.

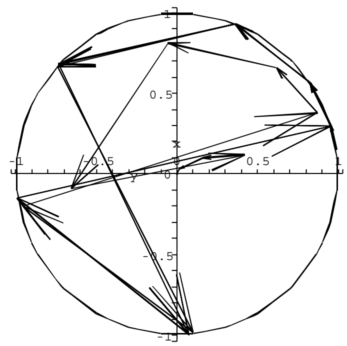

Figure 14: Another orbit of the simplest Julia set with the initial value, $z_0 = cos(\pi/21) + i\, sin(\pi/21)$

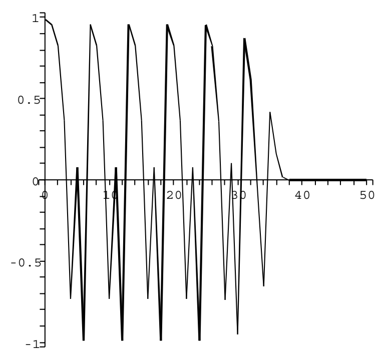

Figure 15: Change in the real-coordinate of the orbit depicted in Figure 3

There is a three cycle in the process shown in figure 17, too. This cycle is composed of the three points, one of which is located at the center of the biggest leaf near the origin, and the other two at the centers of the two biggest leaves in the second quadrant, respectively. Figure 19 shows an orbit which starts from a point of the leaf in a fourth quadrant.

In a more general case, when $p = 1$, eq. (42) can be written as follows,

$$z_{j,n+1} = a_r z_{jn}^r + a_{r-1} z_{jn}^{r-1} + \ldots + z_{jn} + a_0, \tag{50}$$

where the first three factors, a_r, a_{r-1}, and a_{r-2} are defined, respectively, as follows:

$$\begin{aligned} a_r &= (-1)^r f_r(w_{j,n}^{(r)}), \\ f_r(w_{j,n}^{(r)}) &= \sum_{k \neq j}^{N} w_{jk,n}^{(r)}, \end{aligned}$$

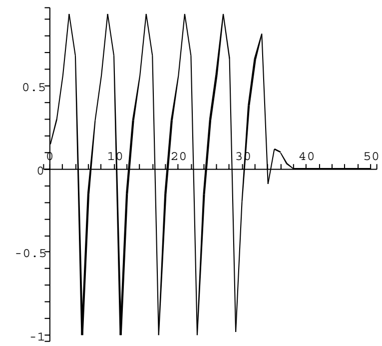

Figure 16: Change in the imaginary-coordinate of the orbit depicted in Figure 3

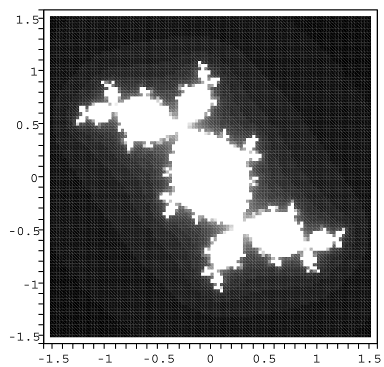

Figure 17: A Mandelbrot process with $c = -0.12 + 0.74i$

$$
\begin{aligned}
a_{r-1} &= (-1)^{r-1} f_{r-1}(w_{j,n}^{(r-1)}), \\
f_{r-1}(w_{j,n}^{(r-1)}) &= \sum_{k \neq j}^{N} w_{jk,n}^{(r-1)} \\
&\quad + \sum_{k \neq j}^{N} {}_rC_{r-1} w_{jk,n}^{(r)} z_{kn},
\end{aligned}
$$

and

$$
\begin{aligned}
a_{r-2} &= (-1)^{r-2} f_{r-2}(w_{j,n}^{(r-2)}), \\
f_{r-2}(w_{j,n}^{(r-2)}) &= \sum_{k \neq j}^{N} w_{jk,n}^{(r-2)} \\
&\quad + \sum_{k \neq j}^{N} {}_{r-1}C_{r-2} w_{jk,n}^{(r-1)} z_{kn}
\end{aligned}
$$

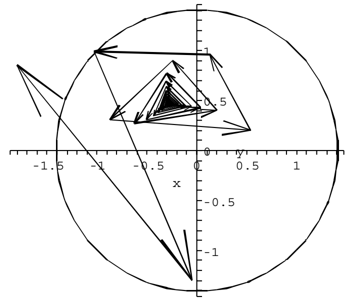

Figure 18: An orbit of the Mandelbrot process shown in Figure 7 with the initial value, $z_0 = -0.2738 + 0.4783i$

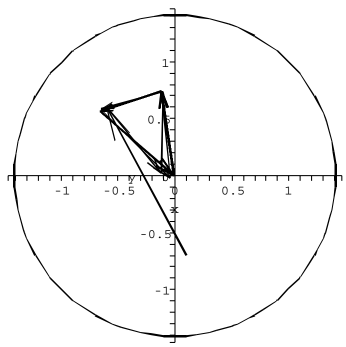

Figure 19: Another orbit of the Mandelbrot process shown in Figure 7 with the initial value, $z_0 = -0.1 - 0.7i$

$$+ \sum_{k \neq j}^{N} {}_{r}C_{r-2} w_{jk,n}^{(r)} z_{kn}^{2}.$$

Finally, factor a_0 is defined as follows:

$$a_0 = \sum_{k \neq j}^{N} \sum_{m=1}^{r} w_{jk,n}^{(m)} z_{kn}^{m}.$$

We have introduced some of the "unusual behaviors" of members in some special cases of our complex difference system model, and have discussed the implications of these abnormal behaviors of members.

These special cases of our general complex difference system model rest on the assumption that a certain member of a group suddenly begins to ignore all the sentiment relationships with the other members from a specified point in time. Such an assumption seems to be unusual, but it seems to be possible that we behave abnormally under some critical situations.

There remains several questions to be answered. One of them may be the question of how long such an abnormal situation might continue working. Another may be the question of what kind of behaviors are predicted for members of a group under a normal situnation. Here, we mean by 'normal' the usual situation in which there is no such a person who ignores all the sentiment relations with other members.

Finally, there exists a fundamental question about our general nonlinear model. Note that equation (42) states that any member of a group is motivated to move basically according to the magnitude of skewness of sentiment between two members including him or her. In other words, this model assumes that members will not move in his or her psychological space if there exists no skewness of sentiment between any two members. However, such an assumption seems to be too restrictive. In the daily life situation our sentiment relatioships between neighbors seem to be dynamic and sometimes flucturating. Considering such a dynamic interpersonal relationships, it seems to be necessary and natural to revise our general model further.

References

Akaike, H. (1973). Information theory and an extension of the maximum likelihood principle. In B. N. Petrov & F. Csaki (Eds.), *The second international symposium on information theory* (pp.267-281). Budapest: Akadémiai Kiado.

Akaike, H. (1977). On entropy maximization principle. In P. R. Krishnaiah (Ed.), *Applications of Statistics* (pp.27-41). Holland: North Holland.

Andersen, E. B. (1980). *Discrete Statistical Models with Social Science Applications*. Amsterdam: North-Holland.

Bishop, Y. M. M., Fienberg, S. E., & Holland, P. W. (1975). *Discrete Multivariate Analysis: Theory and Practice*. Cambridge: MIT Press.

Bowker, A. H. (1948). A test for symmetry in contingency tables. *Journal of the American Statistical Association*, **43**, 572-574.

Caussinus, H. (1965). Contribution à l'analyse statistifque des tableaux de corrélation. *Annales de la Faculté des Sciences de l'Université de Toulouse*, **29**, 77-182.

Chino, N. (1978). A graphical technique for representing the asymmetric relationships between N objects. *Behaviormetrika*, **5**, 23-40.

Chino, N. (1984). Toward a theory of dynamical system in group dynamics. Working paper of the specific researches in education and psychology by the aid of The Ministry of Education in Japan, Nagoya. Nagoya Univ.

Chino, N. (1990). A generalized inner product model for the analysis of asymmetry. *Behaviormetrika*, **27**, 25-46.

Chino, N. (1992). Metric and Nonmetric Hermitian canonical models for asymmetric MDS. *Proceedings of the 20th annual meeting of the Behaviormetric Society of Japan* (pp.246-249), Tokyo, Japan.

Chino, N. (1997). Hitaisho Tajigen-Syakudo Kousei-Ho. [Asymmetric Multidimensio nal Scaling]. Kyoto: Gendai-Sugaku.

Chino, N. (2002). Complex space models for the analysis of asymmetry. In S. Nishisato, Y. Baba, H. Bozdogan, and K. Kanefuji (Eds.) *Measurement and Multivariate Analysis* (pp. 107-114). Tokyo: Springer-Verlag.

Chino, N. (2003a). Complex difference system models for the analysis of asymmetry. In H. Yanai, A. Okada, K. Shigemasu, Y. Kano, and J. J. Meulman (Eds.) *New Developments in Psychometrics* (pp.479-486). Tokyo: Springer-Verlag.

Chino, N. (2003b). Fitting complex difference system models to longitudinal asymmetric proximity matrices. *Paper presented at the 13th International Meeting and the 68th Annual Meeting of the Psychometric Society.* Cagliari, Sardinia, Italy.

Chino, N. (2005). Abnormal behaviors of members predicted by a complex difference system model. *Bulletin of the Faculty of Psychological & Physical Science*, **1**, 69-73.

Chino, N., & Nakagawa, M. (1983). A vector field model for sociometric data. *Proceedings of the 11th annual meeting of the Behaviormetric Society of*

Japan (pp.9-10), Kyoto, September.

Chino, N., & Nakagawa, M. (1990). A bifurcation model of change in group structure. *The Japanese Journal of Experimental Social Psychology*, **29**, No.3, 25-38.

Chino, N., and Shiraiwa, K. (1993). Geometrical structures of some non-distance models for asymmetric MDS. *Behaviormetrika*, **20**, 35-47.

Constantine, A. G. & Gower, J. C. (1978). Graphical representation of asymmetric matrices. *Applied Statistics*, **27**, 297-304.

Cox, T. F. and Cox, M. A. A. (2001). Multidimensional Scaling, 2nd Edition. London: Chapman & Hall/CRC.

De Rooij, M. and Heiser, W. J. (2003). Graphical representations and odds ratios in a distance-association model for the analysis of cross-classified data. *Psychometrika*, **70**, 99-122.

De Rooij, M. and Heiser, W. J. (2005). Graphical representations and odds ratios in a distance-association model for the analysis of cross-classified data. Psychometrika, 70, 99-122.

Escoufier, Y., & Grorud, A. (1980). Analyse factorielle des matrices carrees non symetriques. In E. Diday et al. (Eds.) *Data Analysis and Informatics* (pp.263-276). Amsterdam: North Holland.

Gower, J. C. (1977). The analysis of asymmetry and orthogonality. In J.R. Barra, F. Brodeau, G. Romer, & B. van Cutsem (Eds.), *Recent Developments in Statistics* (pp.109-123). Amsterdam: North-Holland.

Gregerson, H., and Sailer, L. (1993). Chaos theory and its implications for social science research. *Human Relations*, **46**, 777-802.

Guckenheimer, J. & Holmes, P. (1983). *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Berlin: Springer-Verlag.

Harshman, R. A. (1978). Models for analysis of asymmetrical relationships among N objects or stimuli. *Paper presented at the First Joint Meeting of the Psychometric Society and The Society for Mathematical Psychology*, Hamilton, Canada.

Harshman, R. A., Green, P. E., Wind, Y., & Lundy, M. E. (1982). A model for the analysis of asymmetric data in marketing research. *Marketing Science*, **1**, 205-242.

Heath, R. A. (2000). *Nonlinear Dynamics - Techniques and Applications in Psychology*, London: Lawrence Erlbaum Associates.

Hirsch, M. W., & Smale, S. (1974). *Differential equations, dynamical systems, and linear algebra*. New York: Academic Press.

Holland, P. W., & Leinhardt, S. (1977). A dynamic model for social networks. *Journal of Mathematical Sociology*, **5**, 5-20.

Kiers, H. A. L., & Takane, Y. (1994). A generalization of GIPSCAL for the analysis of asymmetric data. *Journal of Classification*, **11**, 79-99.

Krumhansl, C. L. (1978). Concerning the applicability of geometric models to similarity data: The interrelationship between similarity and spatial density. *Psychological Review*, **85**, 445-463.

Kruskal, J. B., & Wish, M. (1978). *Multidimensional scaling*, University paper series on quantitative applications in the social sciences, 07-011. Beverly Hills, California: Sage Publications.

Lewin, K. (1933). Environmental forces. In C. Murchison (Ed.), *A Handbook of child psychology*, Vol.**2**. New York: Russell & Russell. pp.590-625.

Okada, A., & Imaizumi, T. (1987). Nonmetric multidimensional scaling of asymmetric proximities. *Behaviormetrika*, **21**, 81-96.

Okada, A., & Imaizumi, T. (1997). Asymmetric multidimensional scaling of two-mode three-way proximities. *Journal of Classification*, **14**, 195-224.

Peitgen, H.-O, and Richter, P. H. (1986). *The Beauty of Fractals*, New York: Springer-Verlag.

Peixoto, M. C., & Peixoto, M. M. (1959). Structural stability in the plane with enlarged boundary conditions. *Anais da Academia Brasileira de Ciencias*, **31**, 135-160.

Ramsay, J. O. (1977). Maximum likelihood estimation in multidimensional scaling. *Psychometrika*, **42**, 241-266.

Ramsay, J. O. (1982). Some statistical approaches to multidimensional scaling data. *The Journal of the Royal Statistical Society*, Series A (General), **145**, 285-312.

Read, C. B. (1978). Tests of symmetry in three-way contingency tables. *Psychometrika*, **43**, 409-420.

Rocci, R., & Bove, G. (2002). Rotational techniques in asymmetric multidimensional scaling. *Journal of Computational and Graphical Statistics*, **11**, 405-419.

Saburi, S., & Chino, N. (2005). *Proceedings of the 33rd annual meeting of the Behaviormetric Society of Japan* (pp.404-407), Nagaoka, Japan.

Saburi, S., & Chino, N. (2006, submitted). A maximum likelihood method for an asymmetric MDS model.

Saito, T. (1991). Analysis of asymmetric proximity matrix by a model of distance and additive terms. *Behaviormetrika*, **29**, 45-60.

Saito, T., & Takeda, S. (1990). Multidimensional scaling of asymmetric proximity: model and method. *Behaviormetrika*, **28**, 49-80.

Sato, Y. (1988). An analysis of sociometric data by MDS in Minkowski space. In K. Matsusita (Ed.), *Statistical Theory and Data Analysis II* (pp.385-396). Amsterdam: North-Holland.

Takane, Y. (1981). Multidimensional successive categories scaling: A maximum likelihood method. *Psychometrika*, **46**, 9-28.

Tobler, W. (1976-77). Spatial interaction patterns. *Journal of Environmental Systems*, **6**, 271-301.

Torgerson, W. (1952). Multidimensional scaling : I. Theory and method. *Psychometrika*, **17**, 401-419.

Torgerson, W. (1958). *Theory and Methods of Scaling*. New York: Wiley.

Trendafilov, N. T. (2002). GIPSCAL revisited. A projected gradient approach. *Statistics and Computing*, **12**, 135-145.

Weeks, D. G., & Bentler, P. M. (1982). Restricted multidimensional scaling models for asymmetric proximities. *Psychometrika*, **47**, 201-208.

Yadohisa, H., & Niki, N. (1999). Vector firld representation of asymmetric proximity data. *Communications in Statistics - Theory and Method*, **28**, 35-48.

Young, F. W. (1975). An asymmetric Euclidean model for multi-process asymmetric data. Paper presented at U.S.-Japan Seminar on MDS, San Diego, U.S.A.

Young, G., & Householder, A. S. (1938). Discussion of a set of points in terms of their mutual distances. *Psychometrika*, **3**, 19-22.

Zielman, B., & Heiser, W. J. (1993). Analysis of asymmetry by a slide-vector. *Psychometrika*, **58**, 101-114.

Zielman, B., & Heiser, W. J. (1996). Models for asymmetric proximities. *British Journal of Mathematical and Statistical Psychology*, **49**, 127-146.

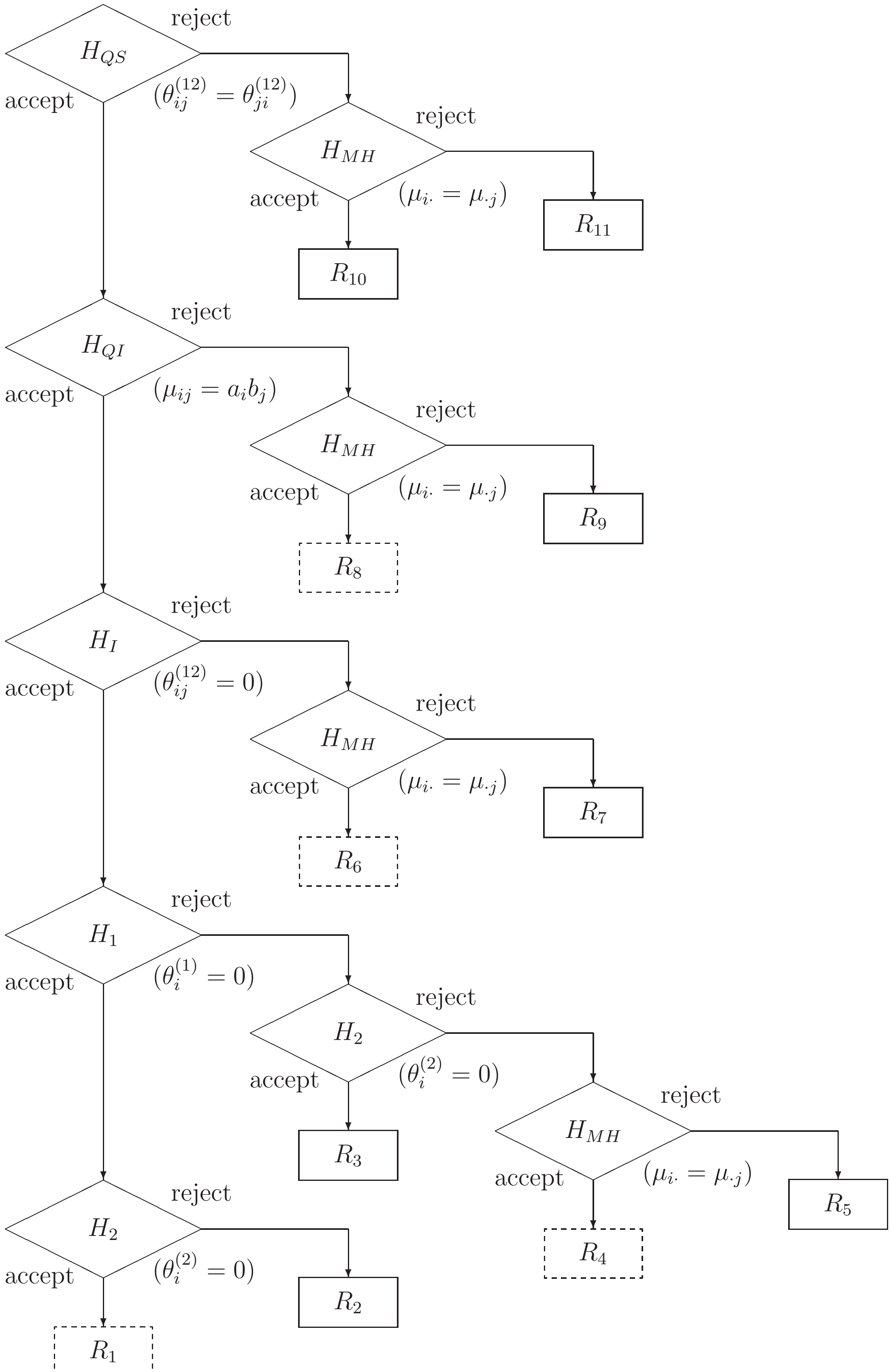

Figure 6: A flow chart of various symmetry tests and a related test in a preprocessing step