An elementary theory of a dynamic weighted digraph (5)

Naohito Chino
Aichi Gakuin University

1 Introduction

This is the **fifth** consecutive report of Chino's (2018a). **In the first report** we proposed an elementary theory of a ***dynamic weighted digraph*** (hereafter abbreviated as ***DWD***) which describes changes in a weighted digraph over time.

As is well known, the weighted digraph is a digraph with weights specified at any time, say, *n*, in which weights are attached to each directed arc (or edge, link) between nodes (or vertices, terminals) as well as each loop of the digraph.

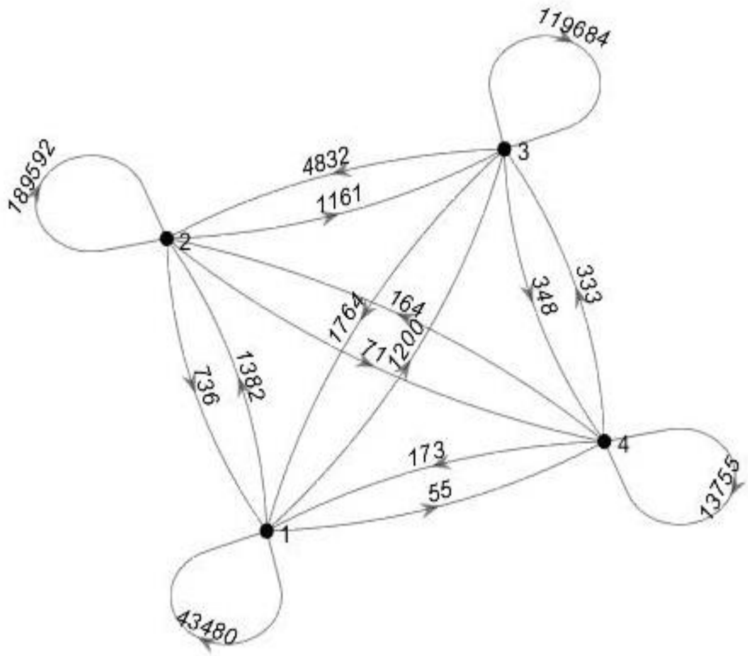

Fig. 1. A weighted digraph **associated with the observed trade data of four nations at time *n***.

Table 1. A trade data among four nations associated with a weighted digraph shown above.

from/to	1.Japan	2.USA	3.China	4.Russia
1.Japan	43,480	1,382	1,200	55
2.USA	736	189,592	1,161	71
3.China	1,764	4,832	119,684	348
4.Russia	173	164	333	13,755

We denote the weight matrix at time n as $W_n = \{w_{jk,n}\}$, $j = 1, \cdots, N$, $k = 1, \cdots, N, n = 1, \cdots, T$, where N is the number of nodes, and T is the number of times.

Dynamic weighted digraph considers changes in a weighted digraph in Fig. 1 over time. This is equivalent to consider changes in an asymmetric similarity data over time in Table 1.

In DWD we assume that there exists **(1)** an **underlying *space of states*** in which nodes of the observed weighted digraph are embedded, and **(2)** the configuration of nodes which varies according to the ***asymmetric interactions*** among nodes, which constitute a constant ***asymmetric interaction matrix* (AIM)**. Here, the underlying space of states in DWD is a finite-dimensional ***Hilbert space*** which is obtained by applying the **Chino and Shiraiwa theorem** (Chino & Shiraiwa, 1993) to the observed asymmetric weight matrix at an initial point in time if the Hermitian matrix associated with this matrix is *positive semi-definite* (i.e., p.s.d.).

Why can we embed nodes of an observed weighted digraph in a Hilbert space? The answer is obtained by considering the eigenvalue problem of the following Hermitian matrix constructed from the observed asymmetric weighted matrix, i.e., asymmetric similarity matrix shown above:

Table 2. The Hermitian matrix associated with the (log transformed) ASM in Table 1.

from/to	1.Japan	2.USA	3.China	4.Russia
1.Japan	10.6801	6.9163 + 0.3150i	7.2827 − 0.1926i	4.5803 − 0.5730i
2.USA	6.9163 − 0.3150i	12.1526	7.7700 − 0.7130i	4.6813 − 0.4186i
3.China	7.2827 − 0.1926i	7.7700 + 0.7130i	11.6926	5.8302 + 0.0220i
4.Russia	4.5803 + 0.5730i	4.6813 + 0.4186i	5.8302 − 0.0220i	9.5292

Here, it is interesting to **compare** the *Young-Householder theorem* on symmetric similarity matrix (Young & Householder, 1938) **with** the *Chino-Shiraiwa theorem* on asymmetric similarity matrix.

Since the latter theorem is lengthy and includes its proof in the original paper

(Chino & Shiraiwa, 1993), we have deleted the proof in the abbreviated theorem:

Theorem (Young and Householder). A necessary and sufficient condition for a set of numbers $d_{jk} = d_{kj}$ to be the *mutual distances* of a real set of points in ***Euclidean space*** is that the matrix $\boldsymbol{B} = \{b_{jk}\}$,

$$b_{jk} = \frac{1}{2}\left(d_{j0}^2 + d_{k0}^2 - d_{jk}^2\right), \tag{a.1}$$

be ***positive semi-definite***; and in this case the set of points is unique apart from a Euclidean transformation. Here, *0* in Equation (a.1) denotes the origin.

Remark. The b_{ij} in Equation (a.1) in the Young-Householder theorem is the ***inner product*** of two vectors in an Euclidean space, and is deduced from the famous ***law of cosines***.

Theorem (Chino and Shiraiwa). A necessary and sufficient condition for a set of numbers $d_{jk} = d_{kj}$ to be the *mutual distances* of a real set of points in a finite-dimensional ***Hilbert space*** is that the matrix $\boldsymbol{H} = \{h_{jk}\}$,

$$h_{jk} = \frac{1}{2}\left(d_{jo}^2 + d_{ko}^2 - d_{jk}^2\right) + \frac{1}{2} i \left(d_{jo}^2 + d_{ko}^2 - \bar{d}_{jk}^2\right), \quad 1 \leq j, k \leq N. \tag{a.2}$$

be ***positive semi-definite***; and in this case the set of points is unique apart from an arbitrary unitary transformation. Here,

$$d_{jk} = \left\|\boldsymbol{v}_j - \boldsymbol{v}_k\right\|, \quad 1 \leq j \leq N, \tag{a.3}$$

$$d_{jo} = \left\|\boldsymbol{v}_j\right\|, \quad 1 \leq j \leq N, \tag{a.4}$$

$$\bar{d}_{jk} = \left\|\boldsymbol{v}_j - i\, \boldsymbol{v}_k\right\|, \quad 1 \leq j, k \leq N. \tag{a.5}$$

and $\boldsymbol{v}_j$ $(1 \le j \le N)$ be a vector of order n in a complex vector space.

First, let us define

$$\boldsymbol{H} = \{h_{jk}\} = \boldsymbol{A}_s + i\boldsymbol{A}_{sk}, \tag{a.6}$$

where $\boldsymbol{A} = \{a_{jk}\} = \boldsymbol{A}_s + \boldsymbol{A}_{sk}$, $\boldsymbol{A}_s = \{(a_{jk} + a_{kj})/2\}$, and $\boldsymbol{A}_{sk} = \{(a_{jk} - a_{kj})/2\}$.

Second, let us solve the eigenvalue problem of $\boldsymbol{H}$ and approximate it using the nonzero eigenvalues of $\boldsymbol{H}$ and the corresponding eigenvectors as $\boldsymbol{H} = \boldsymbol{U}_1 \boldsymbol{\Lambda} \boldsymbol{U}_1^*$. Then, $\boldsymbol{v}_j$ $(1 \le j \le N)$ in the above theorem corresponds to a row vector of order n of $\boldsymbol{U}_1$

Equation (a.2) in the Chino-Shiraiwa theorem is a version of the so-called ***polarization identity*** (or ***polar identity***) which holds in a ***pre-Hilbert space***. The $\boldsymbol{h}_{jk}$ in Equation (a.2) is an ***inner product*** between two points **in a complex space**.

Here, a complex vector space with an inner product is called an inner product space or a ***pre-Hilbert space*** or a unitary space. Therefore, it is apparent that **$\boldsymbol{h}_{jk}$ is a generalization of an inner product $\boldsymbol{b}_{jk}$ in a Euclidean space to a pre-Hilbert space**. It is known that a ***complete*** inner product space is called a ***Hilbert space***.

In DWD, we describe the changes in N nodes over time by a following general set of ***complex difference equations***:

$$z_{j,n+1} = z_{j,n} + \sum_{m=1}^{q} \sum_{k \ne j}^{N} D_{jk}^{(m)} f^{(m)}(z_{j,n} - z_{k,n}) + g(u_{j,n}) + z_0, \quad j = 1,2,\ldots,N, \tag{1}$$

where

$$f^{(m)}(z_{j,n} - z_{k,n}) = \left(\left(z_{j,n}^{(1)} - z_{k,n}^{(1)}\right)^m, \left(z_{j,n}^{(2)} - z_{k,n}^{(2)}\right)^m, \ldots, \left(z_{j,n}^{(p)} - z_{k,n}^{(1p)}\right)^m\right)^t, \tag{2}$$

and

$$D_{jk}^{(m)} = diag\left(\alpha_{jk}^{(1,m)}, \alpha_{jk}^{(2,m)}, \ldots, \alpha_{jk}^{(p,m)}\right). \tag{3}$$

Here, m denotes the degree of the vector function $f^{(m)}(z_{j,n} - z_{k,n})$, which is assumed to have the maximum value q. **Moreover, $\alpha_{jk}^{(1,m)}, \alpha_{jk}^{(2,m)}, \ldots, \alpha_{jk}^{(p,m)}$ are *complex constants*.**

In the second report (Chino, 2019), we discussed the relation between the theory of DWD and that of *evolutionarily stable strategy* (abbreviated as ESS) in biology proposed by Maynard and Price (1973).

In order to compare these two models, we first assumed a special case of DWD when **p=1, m=1, and N=3**. Letting $\boldsymbol{z}_n = (z_{jn}, z_{kn}, z_{ln})^t$, Eq. (1) can be written as

$$\boldsymbol{z}_{n+1} = \boldsymbol{A}_3 \boldsymbol{z}_n, \quad \boldsymbol{A}_3 = \begin{pmatrix} 1 + a_{jk} + a_{jl} & -a_{jk} & -a_{jl} \\ -a_{kj} & 1 + a_{kl} + a_{kj} & -a_{kl} \\ -a_{lj} & -a_{lk} & 1 + a_{lj} + a_{lk} \end{pmatrix}, \tag{4}$$

where all the elements of this matrix are generally **complex numbers**. If the initial configuration of nodes is a **tripartite deadlock**, then the solution curves *diverge* or *converge* depending on the eigenvalues of $\boldsymbol{A}_3$.

For example, if $a_{jk} = -0.01(1-i)$, $a_{jl} = 0.01(1-i)$, $a_{kj} = -0.02(1-i)$, $a_{kl} = 0.02(1-i)$, $a_{lj} = -0.02(1-i)$, and $a_{lk} = 0.02(1-i)$, we have solution curves shown in Fig. a.1.

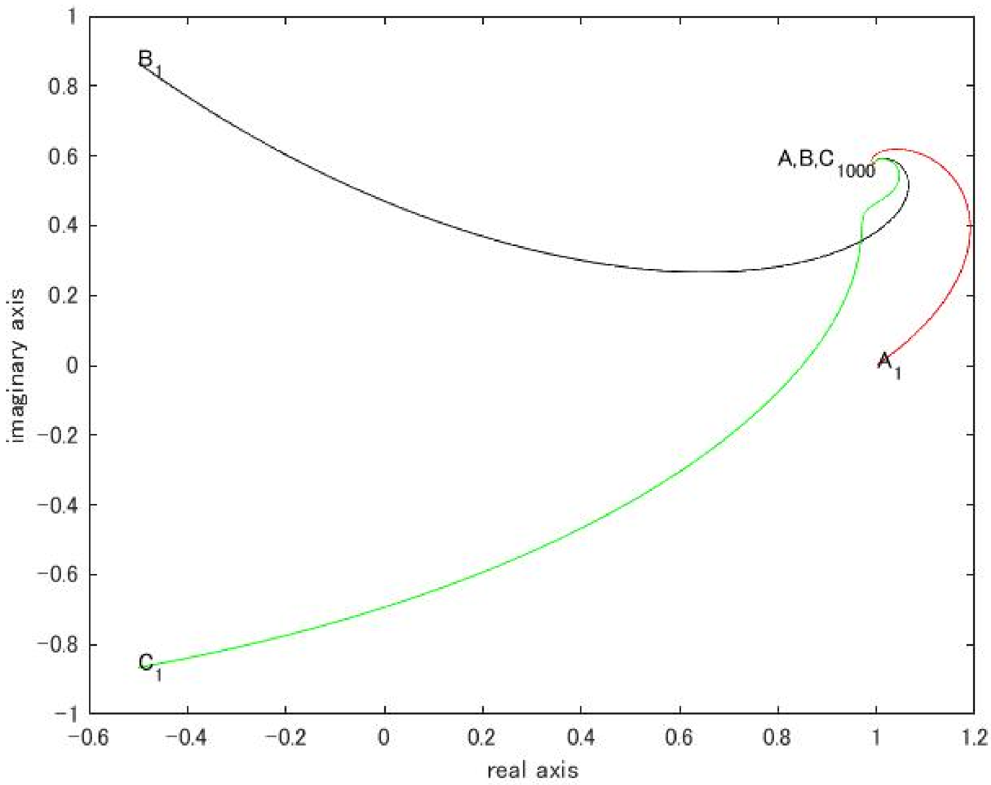

Fig. a.1. Solution curves of Equation (4) with a special case of A_3 (This figure was reproduced from Fig.11 in Chino (2018).

It is apparent from Fig. a.1 that as time proceeds the state of the tripartite deadlock disappears and the **solution curves** of the triad **converge to a point**.

It should be noted here that our general DWD described by Eqs. (1), (2), and (3) enables us to describe various ***chaotic motions*** of nodes even if the space of states is one (complex) dimensional, and the number of nodes is two, **as shown in the next figure (e.g., Chino, 2018b)**.

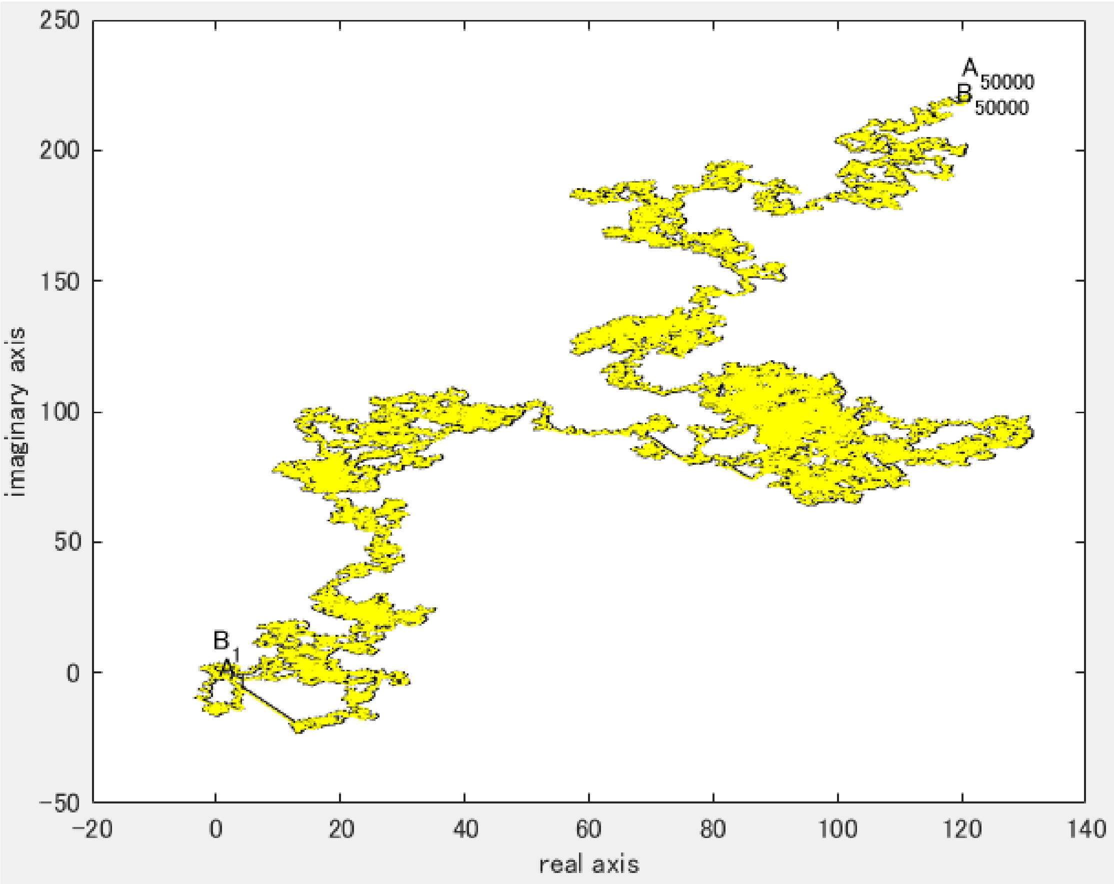

Figure a.2. Simultaneous plot of the trajectories of two nodes after 50,000 iterations.

Next, we introduced a **general two-population model** proposed by Hofbauer (1996), that is

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_j \end{pmatrix} = \begin{pmatrix} x_i\ ((\boldsymbol{Ay})_i - \boldsymbol{x}^t\boldsymbol{Ay} \\ y_j\ ((\boldsymbol{Bx})_j - \boldsymbol{y}^t\boldsymbol{Bx} \end{pmatrix}, \quad \text{i}=\ 1,\dots,\text{n}, \quad \text{j}=\ 1,\dots,\text{m}. \tag{5}$$

where $x = (x_1, \dots, x_n)^t$ is the relative frequency vector for one population, while $y = (y_1, \dots, y_m)^t$ is that for the second population. Moreover, A is the payoff matrix for one population, while B is the payoff matrix for the second population.

Second, we referred to an application of his model to the problem of learning to play with the game of **rock-paper-scissors**, which was investigated by Sato et al. (2002). The payoff matrices of this game are written in general as

$$A = \begin{pmatrix} \varepsilon_x & -1 & 1 \\ 1 & \varepsilon_x & -1 \\ -1 & 1 & \varepsilon_x \end{pmatrix}, \qquad B = \begin{pmatrix} \varepsilon_y & -1 & 1 \\ 1 & \varepsilon_y & -1 \\ -1 & 1 & \varepsilon_y \end{pmatrix}, \tag{6}$$

where $-1 < \varepsilon_x < 1$ and $-1 < \varepsilon_y < 1$. Here, columns of these matrices are placed in the order of "rock", "paper", and "scissors". These are examples of ASMs which are hypothetical.

If $\varepsilon_x = -\varepsilon_y = \varepsilon$, this game is called a *zero sum game.* They found various interesting orbital patterns of system (6) which include those patterns close to **chaotic orbits**, in examining its solution curves when the parameter value ε varies.

In the third report (Chino, 2020), we first pointed out an advantage of DWD over a simple application of asymmetric MDS to an observed ASM.

As is apparent from our DWD formulation discussed above, we distinguish clearly between the ***observed ASM*** *among objects* (stated another way, nodes, elements, and so on) and the ***hypothetical AIM*** among objects. This distinction relieves us of the problem of errors in estimating the configuration of objects, given an observed ASM among objects. Second, we argued the differences between our DWD and some of the **artificial neural network models**. For example, the goal of the associative learning is to determine the weights values for a set of m scalar-valued inputs $\boldsymbol{x} = [x_1, \dots, x_m]^t$ which may be assumed to represent firing frequencies in presynaptic fibers through m synaptic junctions with coupling strengths $\boldsymbol{w} = [w_1, \dots, w_m]^t$ (e.g., Oja, 1982). For example, in the Hebbian rule the coupling strength vector w_n at trial n is renewed as $\boldsymbol{w}_{n+1} = \boldsymbol{w}_n + \eta \boldsymbol{x}_n y(\boldsymbol{x}_n)$, while in Oja's rule as $\boldsymbol{w}_{n+1} = \boldsymbol{w}_n + \eta y_n(\boldsymbol{x}_n - y_n \boldsymbol{w}_n)$. In any case, the coupling strength vector in these models is assumed to change, and the Oja flow, for example, converges finally to a fixed point as learning proceeds.

Second, we took up a method for the **complex-valued multistate Hopfield associative memory** proposed by **Müezzinoğlu et al. (2003)**.

In that method, it is assumed that the network consists of N fully connected neurons, whose states at time instant n constitute the state vector $\boldsymbol{z}[n]$ of the network. Moreover, let w_{ij}

denote the complex-valued weight associated to the coupling from the state of the jth neuron to an input of the ith one. Here, the state of a single neuron, say, lth neuron, at time n is updated according to the following **recurrence equation**:

$$z_l[n+1] = csign_K\left(e^{i(\pi/K)} \sum_j^N w_{lj} z_j[n]\right), \quad l = 1, \cdots, m, \tag{6}$$

while keeping all other states unchanged. Here, $csign_k(u)$ is defined as e^0 if $0 \leqq arg(u) < \frac{2\pi}{K}$, $e^{i2\pi/K}$ if $\frac{2\pi}{K} \leqq arg(u) < \frac{4\pi}{K}$, …, and $e^{i[2\pi/K]/(K-1)}$ if $(K-1)\frac{2\pi}{K} \leqq arg(u) < 2\pi$, where K is the resolution factor of the network, and it determines the cardinality of the finite state space, while m is the order of the state vector z.

Here, $e^0, e^1, \ldots, e^{K-1}$ are the **primitive *K*th power root of a unit**, and K points on the unit circle in the complex plane, and K points on the unit circle in the complex plane (Aizenberg & Aizenberg, 1992; Jancowski et al., 1996; Noest, 1988). In this model, iterations are continued until the following *computational energy function* reaches a minimum:

$$E(\boldsymbol{z}) = -\frac{1}{2}\boldsymbol{z}^H \boldsymbol{W} \boldsymbol{z}. \tag{7}$$

A sufficient condition on the convergence of the recursion (6) has been reported in Jancowski et al. (1996) as a Hermitian weight matrix ($\boldsymbol{W} = \boldsymbol{W}^H$) with nonnegative diagonal entries.

In the fourth report (Chino, 2021), we pointed out that our method for embedding an observed or hypothetical weight matrix to a Hilbert space is clearly an extension of the spectral graph theory which has been extensively studied in various disciplines of science (e.g., Chung, 1997; Brouwer & Haemers, 2012).

In this report, we shall discuss the difference in model between DWD and traditional dynamical system models for asymmetric interactions among objects or nodes.

2 Difference between DWD and dynamical system models for asymmetric interactions

As regards the dynamical system models for asymmetric interactions, a variety of systems of difference and differential equations have been proposed. Some of them are based on the *established laws* which govern the phenomena in each of the discipline of science, while others are not.

For example, in ***electrical circuit*** there exist a few laws such as the *Ohm's law* and *Faraday's law* which govern the current and voltage in the circuit.

As a result, the system of differential equations which describes the changes in the state of the circuit over time is determined uniquely as a two-dimensional system whose state variables are the current through the inductor and the voltage across the capacitor branch (Hirsch & Smale, 1974).

One is the ***generalized Ohm's law***,

$$f(i_R) = v_R, \tag{a.7}$$

which describes the functional relationship between the current i_R through the resister v_R. Another is the ***Faraday's law***,

$$L\frac{di_L(t)}{dt} = v_L(t), \tag{a.8}$$

which describes the change in the current i_L through the inductor over time. The other is the **condition imposed on the capacitor *C***,

$$C\frac{dv_C(t)}{dt} = i_C(t), \tag{a.9}$$

which describes the change in the voltage at the capacitor over time. Finally, these five equations can be simplified, considering the relations between them, as

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} y - f(x) \\ -x \end{pmatrix}, \tag{a.10}$$

where $x = i_L$ and $y = v_C$. Hersch and Smale (op. cit., p.214) discuss the derivation of (a.9) in some detail.

Another example is a ***chemical reaction network***. This network is basically governed by the *law of mass action*, and the fundamental form of the system of differential equations which describes the change in the concentrations of the chemical species over time is written as a system of polynomial differential equations.

However, in most of the networks mainly observed in social and behavioral sciences, it is rare that such strict laws have been established. Therefore, in such cases it will be necessary and appropriate to consider general system models which possibly explain various phenomena. In this sense, DWD may be said to meet this requirement.

Finally, most of the extant dynamical system models for asymmetric interactions assume implicitly or explicitly that the state space is a ***Euclidean space***, while DWD assumes that it is a ***Hilbert space*** **or** ***indefinite metric space***, depending on the definiteness of the Hermitian matrix constructed uniquely from an observed ASM.

It should be noticed here that a Hilbert space includes a Euclidean space as a special case. We shall discuss in more detail at the conference.

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chino@dpc.agu.ac.jp