Extension of Stevens’ definitions of the four scales to the multidimensional case

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1. **Introduction**

As is well known, Stevens (1946, 1951) published two famous papers on the theory of scales of measurements, in which he showed that there are four scale levels of measurements and four corresponding *group structures* of *group theory* *in mathematics*. However, the *mathematical transformations* associated with these group structures were limited to the *unidimensional case* in Stevens papers.

 After only a few years, Torgerson (1954, 1958) proposed the method of multidimensional scaling (classical MDS). Nowadays, MDS is extended to asymmetric MDS (e.g., see Chino, 2012). Therefore, it seems to be necessary to extend the mathematical group structures and the mathematical transformations to the multidimensional case and asymmetric similarity data, respectively.

 Several papers have expanded Stevens’ theory to the multidimensional case (e.g., Adams et al., 1965; Chiang, 1995, 1997, 2001, 2005; Velleman et al., 1993). However, most focus on the invariance of multivariate statistics, rather than addressing the two key theorems on the metric structures of multidimensional scales, the Young-Householder theorem (1938) in symmetric MDS and the Chino-Shiraiwa theorem (1993) in asymmetric MDS. These theorems provide necessary and sufficient conditions for embedding objects in metric spaces. Therefore, the emphasis on statistical invariance seems to be secondary to the more critical concern of examining the invariance of the metric properties of the constructed multidimensional scales, particularly in psychological scaling.

1. **Mathematical transformations associated with the four scales in Stevens’ papers and their extensions to the multidimensional case**

Mathematical group structures and the corresponding transformations listed in Table 1 and Table 6 of Stevens (1946) and Stevens (1951), respectively, are as follows (Chino, 2024):

1. For the *nominal scale*, the corresponding mathematical group is the *permutation group*, and the transformation associated with this group is written as

$x^{'}=f\left(x\right)$, (1)

 where $f\left(x\right)$ means any one-to-one substitution.

1. For the *ordinal scale*, the corresponding mathematical group is the *isotonic group*, and the transformation associated with this group is written as

$x^{'}=f\left(x\right)$, (2)

 where $f\left(x\right)$ means any *monotonic increasing function*.

1. For the *interval scale*, the corresponding mathematical group is the *general linear* *group*, and the transformation associated with this group is written as

$x^{'}=ax+b$, (3)

1. For the ratio scale, the corresponding mathematical group is the *similarity* *group*, and the transformation associated with this group is written as

$x^{'}=ax$, (4)

 As regards the transformation associated with the permutation group, it will be

unnecessary to extend it to the multidimensional case. The transformation

associated with the isotonic group is sometimes called *isotonic transformation*

(e.g., Kim and Kozhukhov, 2007).

As for the transformation described by Eq. (3), we can generalize it to the multidimensional case as

 $\left[\begin{matrix}1\\x\_{1}^{'}\\\begin{matrix}\vdots \\x\_{n}^{'}\end{matrix}\end{matrix}\right]=\left[\begin{matrix}1&0&\begin{matrix}\cdots &0\end{matrix}\\b\_{1}&a\_{11}&\begin{matrix}\cdots &a\_{1n}\end{matrix}\\\begin{matrix}\vdots \\b\_{n}\end{matrix}&\begin{matrix}\vdots \\a\_{n1}\end{matrix}&\begin{matrix}\begin{matrix}\cdots \\\cdots \end{matrix}&\begin{matrix}\vdots \\a\_{nn}\end{matrix}\end{matrix}\end{matrix}\right]\left[\begin{matrix}1\\x\_{1}\\\begin{matrix}\vdots \\x\_{n}\end{matrix}\end{matrix}\right]$, (5)

or equivalently,

 $x^{'}=Ax+b$,(6)

where ***A*** is a square matrix of order *n*,which is *nonsingular*, that is,

 $A=\left[\begin{matrix}a\_{11}&\cdots &a\_{1n}\\\vdots &\cdots &\vdots \\a\_{n1}&\cdots &a\_{nn}\end{matrix}\right]$, (7)

and $x^{'}=\left[\begin{matrix}x\_{1}^{'}&\cdots &x\_{n}^{'}\end{matrix}\right]^{t}$, $b=\left[\begin{matrix}b\_{1}&\cdots &b\_{n}\end{matrix}\right]^{t}$.

According to Horn and Johnson (1985), nonsingular matrices in $M\_{n}\left(F\right)$ such as ***A*** defined by Eq. (7) form a *general linear group*, often denoted $GL\left(n,F\right)$. As a result, the matrix of the first right-hand term of Eq. (5) can be said to form another general linear group of order $n+1$, which may be denoted by $GL\left(n+1,F\right)$. It is not difficult to show that this matrix is also nonsingular.

Eq. (6) is called an *affine transformation*, and the first part of Eq. (5) forms an

*affine group* (e.g., Bloom, 1979). Therefore, we might also say that a general linear group defined by Stevens can be replaced by (a special case of) an affine group rather than a general linear group.

 In matrix theory, it is well known that the *similarity transformation*

 $AX=X diag\left(λ\_{i}\right)$, or $X^{-1}AX=diag\left(λ\_{i}\right)$ , (8)

reduces the matrix ***A*** to diagonal form, if the eigenvalues of ***A*** are distinct (e.g.,

Wilkinson, 1965). If we rewrite the first part of Eq. (8) as

 $x^{'}=Ax (=λx)$, (9)

it is thought of as an extension of Eq. (4) in the *n*-dimensional case. The simlari-

ty transformations in an n-dimensional Euclidean space form a *similarity (trans-*

*formation) group*, $G\_{E}^{n}$ (e.g., Kawada, 1976; Milne, 2017).

In symmetric MDS, configuration of objects is obtained assuming that it is

defined in some real metric space such as a *Euclidean space*. Of course, the necessary and sufficient condition for a set of numbers $d\_{jk}=d\_{kj}$ to be the mutual distances of a real set of points in Euclidean space is that the matrix $B=\left\{b\_{jk}\right\}$,

 $b\_{jk}=\left(d\_{j0}^{2}+d\_{k0}^{2}-d\_{jk}^{2}\right)/2$, (9’)

be *positive semi-definite* and in this case the set of points is unique apart from a Euclidean transformation. Here, *0* in (9’) denotes the origin.

In most of the asymmetric MDS models which are called *augmented distance models*, however, special care should be taken because there is a possibility that the metric assumption is not necessarily fulfilled by these models including the symmetric and asymmetric parts except for the two-dimensional case. For further details see Appendix 1.

By contrast, in HFM proposed by Chino and Shiraiwa (1993) in asymmetric MDS, a *holistic metric structure* including these two parts is defined in a (complex) *Hilbert space*. A Hilbert space is an extension of a Euclidian space to a complex space. In HFM, the observed similarity matrix ***S*** measured at a ratio scale level is decomposed into symmetric part and the skew-symmetric part, first, and then a *Hermitian matrix* $H={\left(S+S^{t}\right)}/{2}+i{\left(S-S^{t}\right)}/{2}$ is constructed. Finally, a *unitary similarity transformation* is applied to ***H*** such that

 $HU=UD$ or $U^{-1}HU=D$, (10)

where ***D*** is a diagonal matrix. If we rewrite the left part of Eq. (10) as

 $u^{'}=Hu\left(=λu\right)$, (11)

This is thought of as an extension of Eq. (9) in the similarity transformation. Generally, the unitary similarity transformations form a *unitary (similarity) group*, $U\left(n\right)$ (e.g., Arnold, 1978).

 The notion of unitary similarity can be generalized to that of *congruence*. Two matrices ***A*** and ***B*** are said to be *congruent* if there exists a nonsingular matrix ***P*** such that $A=PBP^{\*}$. It is well known that a Hermitian matrix$H\in M\_{n}\left(F\right)$ is congruent to the matrix

 $D\_{0}=\left[\begin{matrix}I\_{s}&O&O\\O&-I\_{r-s}&O\\O&O&O\_{n-r}\end{matrix}\right]$, (12)

where *r* = rank ***H*** and *s* is the number of positive eigenvalues of ***H*** counted according to multiplicities, $I\_{t}$ , $t=s, r-s$, is an identity matrix of order *t*, and ***O’*** s are zero matrices of specified orders (e.g., Lancaster & Tismenetsky, 1985, p.184). The *positive semi-definiteness* of ***H*** constructed from the original similarity matrix ***S***, which means that ***H*** is composed only of $I\_{r}$ and $O\_{n-r}$, is the necessary and sufficient conditions for *n* objects to be embedded in a Hilbert space (Chino & Shiraiwa, op. cit.). By contrast, if ***H*** is of the form described by Eq. (12), *n* objects can be embedded in an *indefinite-metric space.*

Considering the above result on the properties of the ratio scale in the asymmetric similarity data, it will be appropriate and necessary to add a unitary group and the corresponding transformations, that is, the *unitary similarity transformations,* to the summary tables of scales of measurement by Stevens (1946, 1951)*.*

1. **Properties of the metric spaces**

In the case of a unidimensional scale, the property of the metric space had been restricted to the constancy of the ratio of a distance between any two points to a distance between another two points in the scale. However, in the case of a multidimensional scale, we must refer to the properties of the metric space spanned by a set of eigenvectors of ***A*** in Eq. (9), one example of which is a *covariance matrix* in PCA. Since a covariance matrix is a *Gram matrix* $XX^{t}$computed from the data matrix ***X*** in $M\_{n,N}\left(R\right)$, we can embed variables in a specified dimensional *Euclidean space*. This condition is equivalent to that of the Euclidean representation of a graph based on its Laplacian matrix (e.g., Brouwer & Haemers, 2012, p.105). On the other hand, the Young-Householder theorem in psychometrics is an alternative condition to Brouwer & Haemers’. Likewise, in HFM the Hermitian matrix ***H*** computed from an asymmetric (real) similarity data matrix ***S*** in $M\_{N,N}\left(C\right)$ is a Gram matrix (Lancaster & Tismenetsky, 1985, p.110) and we can embed objects in a specified dimensional *Hilbert space* if ***H*** is positive semi-definite by Chino and Shiraiwa theorem. A Hilbert space is also an inner product space so that several fundamental properties are known such as *triangle inequality*, *parallelogram law*, *Pythagorean Formula*, and *polar identity* which is a generalization of the *cosine law* in a Euclidean space (e.g., Chino, 1998). For further details see Appendix 2.

**Appendix 1**.

 In this appendix we shall discuss the procedure for checking whether the traditional augmented (distance) models in asymmetric MDS fulfill the holistic metric assumption which includes both the symmetric part and the asymmetric part and show some results of applications of this procedure to some augmented (distance) models.

 First, the procedure consists of the following two steps:

*Step 1*.

 Recover the estimated asymmetric similarity matrix (ASM) by utilizing a p-dimensional estimated configuration or hypothesized configuration of objects of the augmented model under study. In the case of an inner product model (i.e., an augmented model) like Chino’s ASYMSCAL and GIPSCAL or a “pseudo” inner product model such as DEDICOM, we can recover such an ASM by directly computing the estimated ASM from the model under study. On the other hand, in the case of an augmented distance model such as Okada-Imaisumi model, the distance-density model, Weeks-Bentler model and so on, compute the estimated ASM using the estimated configuration or hypothesized configuration of objects obtained via the augmented distance model under study. It should be noticed that in this case the estimated ASM is not composed of the distances between objects but is composed of the real inner products of objects computed based on the estimated configuration of objects via the augmented distance model under study. Moreover, the diagonal elements of the estimated ASM should be computed from the estimated configuration of objects. To do so, we must compute two kinds of real inner products using the upper triangular and the lower triangular parts of the estimated distance matrix.

*Step 2*.

 Apply HFM to the estimated ASM in Step 1 and solve the eigenvalue problem of the Hermitian matrix corresponding to the ASM. If the Hermitian matrix obtained is positive semi-definite, then the holistic metric structure of the augmented model under study has a Hilbert space structure, while if the Hermitian matrix is indefinite, then the augmented model cannot have the holistic metric structure.

**Example 1**.

 In the case of DEDICOM, we used the estimated 3-dimensional configuration of a car-switching data by Harshman et al. (1982). The obtained eigenvalues of the Hermitian matrix were 1.0651, 0.3680, and －0.0199. This shows that the holistic metric structure of the 3-dimensional solution of DEDICOM is indefinite, which means that the DEDICOM’s 3-dimensional solution has not a holistic metric structure. However, if we approximate it using a one-dimensional configuration of HFM corresponding to the largest eigenvalue, then we can embed it in a complex plane (i.e., one-dimensional Hilbert space) where a metric structure is fulfilled.

**Example 2**.

 In the case of **Okada-Imaizumi model**, we used their estimated 3-dimensional configuration of a car-switching data by Harshman et al. (1982), as in Example 1.

The obtained eigenvalues of the Hermitian matrix were 9.5660, 5.0008, and 3.4369. This shows that the holistic metric structure of the 3-dimensional solution of the Okada-Imaizumi model has a Hilbert space structure.

By contrast, if we apply HFM directly to a car-switching data which is a modified data of Harshman et al. (1982) by Okada-Imaizumi, then the eigenvalues of the Hermitian matrix were 2.84, 1.46, 1.30. This means that the holistic metric structure has also a Hilbert space structure.

However, if we apply a log transformation to their car-switching data first and apply HFM to this set of data, then the eigenvalues of the Hermitian matrix were 131.34, 19.08, －8.16, which means that in this case the obtained three-dimension al configuration no longer has a Hilbert space structure.

**Example 3**.

 In Example 3, we shall examine whether the hypothetical configuration of Chino’s ASYMSCAL (Chino, 1978, Fig.3) has a holistic metric structure other than the Euclidean structure of its symmetric part. To check it we computed the Hermitian matrix using the hypothetical configuration. The largest eigenvalue of this matrix was 5.25 and the remaining eigenvalues are all zeros. This shows that in the two-dimensional case, which is equivalent to the one-dimensional complex case, the configuration has a holistic Hilbert space structure except for the positive direction of the configuration is clockwise.

Example 4.

 In this example, we shall examine whether the hypothetical 3-dimensional configuration of GIPSCAL (Chino, 1990) has a holistic metric structure other than the Euclidean structure in the Euclidean structure of its symmetric part, where the hypothetical configuration and the $I^{\*}$ matrix which is peculiar to GIPSCAL in the 3-dimensional case were set to

$X=\left[\begin{matrix}0.0&1.0&-1.0\\-1.0&0.0&1.0\\\begin{matrix}0\\1.0\\\begin{matrix}0.7\\0.0\end{matrix}\end{matrix}&\begin{matrix}-1.0\\0.0\\\begin{matrix}0.7\\0.5\end{matrix}\end{matrix}&\begin{matrix}0.0\\-1.0\\\begin{matrix}0\\1.0\end{matrix}\end{matrix}\end{matrix}\right]$, $I^{\*}=\left[\begin{matrix}0&1&-1\\-1&0&1\\1&-1&0\end{matrix}\right]$.

The eigenvalues of the Hermitian matrix constructed from the original 3-dimensional configuration were 9.8368, －2.1138, 1.5270, 0.0, 0.0, and 0.0. This shows that the 3-dimensional GIPSCAL solution has a holistic indefinite metric structure.

**Appendix 2**.

 In this appendix we shall discuss some laws and formulas on the metric space. Before proceeding, we shall define two important notions, that is, an abstract vector space and a norm of a vector space.

 **Definition 1**. An abstract vector space over a field *F* is a nonempty set *V* of elements called vectors together with operations called vector addition and scalar multiplication. Here, *F* may be real or complex. For further detail, see Chino (op. cit.).

 **Definition 2**. A norm on a vector space *V* is a real-valued function that associates with every element **x** in *V* a quantity $\left‖x\right‖$ (the norm of **x**) such that for all scalars $λ$ and all vectors **x** and **y**,

1. $\left‖x\right‖>0$, if $x\ne 0$**, 2.** $\left‖λx\right‖$=$\left|λ\right|\left‖x\right‖$, 3. $\left‖x+y\right‖\leq \left‖x\right‖+\left‖y\right‖$ (triangle inequality).

Then, we have the following theorems:

**Theorem 1** (Parallelogram Law). For any two elements ***x*** and ***y*** of an (real or complex) inner product space we have

 $\left‖x+y\right‖^{2}+\left‖x-y\right‖^{2}$=2$\left(\left‖x\right‖^{2}+\left‖y\right‖^{2}\right)$ (13)

**Theorem 2** (Pythagorean Formula). For any pairs of orthogonal vectors ***x*** and ***y*** in an (real or complex) inner product space we have

 $\left‖x+y\right‖^{2}=\left‖x\right‖^{2}+\left‖y\right‖^{2}$. (14)

**Theorem 3** (Polar Identity).

 $\left〈x,y\right〉=\frac{1}{2}\left(\left‖x\right‖^{2}+\left‖y\right‖^{2}-\left‖x-y\right‖^{2}\right)+\frac{1}{2}i\left(\left‖x\right‖^{2}+\left‖iy\right‖^{2}-\left‖x-iy\right‖^{2}\right)$, (15)

which is equivalent to

 $\left〈x,y\right〉=\frac{1}{2}\left(\left‖x\right‖^{2}+\left‖y\right‖^{2}-\left‖x-y\right‖^{2}\right)+\frac{1}{2}i\left(\left‖x\right‖^{2}+\left‖y\right‖^{2}-\left‖x-iy\right‖^{2}\right)$. (16)

Proofs of the above theorems are shown, for example, in Debnath and Mikusi**ński (1990, pp.90-92)**.

**Remark**.

From Equation (16), it is apparent that we must compute two kinds of the distances between vectors ***x*** and ***y***, that is,

 $d\_{xy}^{2}=\left‖x-y\right‖^{2}$ and $\overbar{d}\_{xy}^{2}=\left‖x-iy\right‖^{2}$ , (17)

in computing the complex inner product $\left〈x,y\right〉$.

Equation (15) is an extension of the Law of Cosines in a real vector space. Here, the Law cosines in a Euclidean space is defined as the first right-hand side of Equation (15) because it can be deduced from the familiar Law of Cosines in a Euclidean space:

 $\left‖x-y\right‖^{2}=\left‖x\right‖^{2}+\left‖y\right‖^{2}-2\left‖x\right‖\left‖y\right‖ cos\left〈x,y\right〉$. (16)

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