A general non-Newtonian n-body problem and dynamical scenarios of solutions

Naohito Chino
Department of Psychology, Aichi Gakuin University

Paper presented at the 31th International Congress of Psychology,
July 29, 2016
Yokohama, Japan

The organization of my talk

(1) The purpose of this study
(2) Why a Hilbert space model?
(3) Nonlinear difference equation model
(4) Possible dynamical scenarios of solutions
(5) Relations to dissipative structure and so on

(1) The purpose of this study

The purpose of this study is to predict changes in asymmetric relationships among members of a group over time. Members may be various objects which are observed in various branches of sciences, such as nations, human beings and other animals, cells, and so on. A typical example of such data in psychology is a set of longitudinal sociomatrices gathered by Newcomb (1961). Recently, we have been developing **difference equation models** which describe and predict changes in these asymmetric relationships (2000, 2002, 2006, 2014, 2015a, b).

(2) Why a Hilbert space model

In our difference equation models, the **state space** is assumed to be a p-dimensional *Hilbert space* or a p-dimensional *indefinite metric space.* The reason why we choose these spaces comes from a fundamental theorem on the **asymmetric multidimensional scaling** (abbreviated as asymmetric MDS) developed by Chino and Shiraiwa (1993).

Theorem (*Chino and Shiraiwa*)

Let **S** be an N by N asymmetric similarity data matrix among N members of a group (measured at a ratio level), and let **H** be a *Hermitian matrix* computed from the data matrix **S** such that $\mathbf{H}=(\mathbf{S}+\mathbf{S}^t)/2 + i\,(\mathbf{S}-\mathbf{S}^t)/2$, where i is a *pure imaginary number.* Then, a necessary and sufficient condition for the members to be expressible in terms of (complex) Hilbert space is that **H** is *positive semi-definite*.

This theorem tells us that any asymmetric relationships among members of a group at any time can be embedded in one of those spaces depending on the eigenvalues of **H**. The asymmetric MDS model based on this theorem is called the *Hermitian Form Model* (abbreviated as **HFM**, later).

(3) Nonlinear difference equation model

It should be noticed here that the above theorem merely specifies the appropriate state space in which members of a group are located **at an instant of time**. In contrast, the goal of our study is to predict changes in asymmetric relationships among members of a group **over time**, given a set of longitudinal asymmetric relationship data matrices, $\mathbf{S}_1$, $\mathbf{S}_2$, …, $\mathbf{S}_T$. Here, $\mathbf{S}_n$ denotes the asymmetric relational data matrix at time n.

Since the above theorem provides us with the configuration of members on the space at each time, we can obtain a set of longitudinal configuration matrices, $\mathbf{Z}_1$, $\mathbf{Z}_2$,

…, $\mathbf{Z}_T$. Here, let us denote $\boldsymbol{z}_{j,n}$ as the p-dimensional coordinate vector of member j at time n. If the asymmetric relational data is measured at the interval level or ordinal level, we can obtain each of the longitudinal configuration matrices by an *asymmetric maximum likelihood MDS* developed by Saburi and Chino (2008).

We need some appropriate model which predicts changes in these coordinate vectors of each member over time. Candidates for such a model may be **nonlinear difference equation models**.

Model 1

A general model which we consider as *the most fundamental one* may be the following *complex* nonlinear difference equation model:

$$\boldsymbol{z}_{j,n+1} = \boldsymbol{z}_{j,n} + \sum_{m=1}^{q} \sum_{k \neq j}^{N} \boldsymbol{D}_{jk,n}^{(m)} \boldsymbol{f}^{(m)}\left(\boldsymbol{z}_{j,n} - \boldsymbol{z}_{k,n}\right), \; j = 1, 2, \cdots, N, \quad (1)$$

$$\boldsymbol{f}^{(m)}\left(\boldsymbol{z}_{j,n} - \boldsymbol{z}_{k,n}\right) = \left(\left(z_{j,n}^{(1)} - z_{k,n}^{(1)}\right)^m, \left(z_{j,n}^{(2)} - z_{k,n}^{(2)}\right)^m, \ldots, \left(z_{j,n}^{(p)} - z_{k,n}^{(p)}\right)^m\right)^t, \quad (2)$$

and

$$\boldsymbol{D}_{jk,n}^{(m)} = diag\left(w_{jk,n}^{(1,m)}, w_{jk,n}^{(2,m)}, \ldots, w_{jk,n}^{(p,m)}\right). \quad (3)$$

Here, $\boldsymbol{z}_{j,n}$ denotes the coordinate vector of member j at time n in a p-dimensional Hilbert space or a p-dimensional indefinite metric space. Moreover, m denotes the degree of the vector function $\boldsymbol{f}^{(m)}\left(\boldsymbol{z}_{j,n} - \boldsymbol{z}_{k,n}\right)$, which is assumed to have the maximum value q.

Moreover, $w_{jk,n}^{(1,m)}, w_{jk,n}^{(2,m)}, \ldots, w_{jk,n}^{(p,m)}$ are *complex constants*. This model is very general and might enable us to describe various possible changes in asymmetric relationships among members over time.

Model II

Another type of model is a real version of the above complex difference equation model. In general, any p-dimensional Hilbert space as well as indefinite metric space may be viewed as a *2p*-dimensional Euclidean space. As a result, this type of model can be said to be a real difference equation model. In this case, we assume more specific changes in asymmetric relationships among members. That is, we assume that the members obey the following three basic principles of interpersonal behaviors (e.g., Chino, 2015a, b):

1. The asymmetric sentiment relationships among members make their affinities change.
2. If a member has a *positive sentiment* toward another member, then he or she *moves toward* the target member.
3. If a member has a *negative sentiment* toward another member, then he or she *moves away from* the target member.

At this point, it should be noticed that according to the real version of HFM, similarity of member *j* to member *k* at a given time in a two-dimensional Euclidean space can be written as

$$s_{jk} = \lambda(x_{j1}x_{k1} + x_{j2}x_{k2}) + \lambda(x_{j2}x_{k1} - x_{j1}x_{k2}),$$
$$= |\boldsymbol{x}_j||\boldsymbol{x}_k|(\cos\theta_{jk} - \sin\theta_{jk}), \quad (4)$$

in the case when arg $\boldsymbol{x}_k$ is greater than arg $\boldsymbol{x}_j$, where arg $\boldsymbol{x}$ is the angle which the vector originated from the origin to $\boldsymbol{x}$ makes with the positive x-axis, and is assumed to take values ranging from 0 to π.

Moreover, in a two-dimensional Euclidean space, Eq. (1) reduces to the following simple form in the case when q equals 1:

$$\boldsymbol{x}_{j,n+1} = \boldsymbol{x}_{j,n} + w_{jk}(\boldsymbol{x}_{j,n} - \boldsymbol{x}_{k,n}),\ \boldsymbol{x}_{k,n+1} = \boldsymbol{x}_{k,n} + w_{kj}(\boldsymbol{x}_{k,n} - \boldsymbol{x}_{j,n}), \quad (5)$$

where $\boldsymbol{x}_{j,n}$ is the coordinate vector of member *j* at time *n* in the two-dimensional Euclidean space.

If members obey the *basic principles assumed above*, then the signs of w_{jk} and w_{kj} corresponding to those of s_{jk} and s_{kj}, respectively, must be chosen appropriately (Chino, 2016).

<u>*Model III*</u>

A third type of model is the model in which $w_{jk,n}^{(l,m)}$ is specified further in Model *I*. That is,

$$\boldsymbol{z}_{j,n+1} = \boldsymbol{z}_{j,n} + \sum_{m=1}^{q} \sum_{k \neq j}^{N} \boldsymbol{D}_{jk,n}^{(m)} \boldsymbol{f}^{(m)}(\boldsymbol{z}_{j,n} - \boldsymbol{z}_{k,n}),\ j = 1, 2, \cdots, N, \quad (1)$$

$$\boldsymbol{f}^{(m)}(\boldsymbol{z}_{j,n} - \boldsymbol{z}_{k,n}) = \left(\left(z_{j,n}^{(1)} - z_{k,n}^{(1)}\right)^m,\ \left(z_{j,n}^{(2)} - z_{k,n}^{(2)}\right)^m, \ldots, \left(z_{j,n}^{(p)} - z_{k,n}^{(p)}\right)^m \right)^t, \quad (2)$$

and

$$\boldsymbol{D}_{jk,n}^{(m)} = diag\left(w_{jk,n}^{(1,m)}, w_{jk,n}^{(2,m)}, \ldots, w_{jk,n}^{(p,m)}\right). \quad (3)$$

$$w_{jk,n}^{(l,m)} = a_n^{(l,m)} r_{j,n}^{(l,m)} r_{k,n}^{(l,m)} \sin\left(\theta_{k,n}^{(l,m)} - \theta_{j,n}^{(l,m)}\right),$$
$$l = 1, 2, \ldots, \mathrm{p}, \quad m = 1, 2, \ldots, \mathrm{q}. \quad (6)$$

Here, $a_n^{(l,m)}$ is a real constant, and $r_{j,n}^{(l,m)}$ and $\theta_{j,n}^{(l,m)}$ are, respectively, the norm and the argument of $\boldsymbol{z}_{j,n}$ at time *n* on dimension *l*. As a result, $w_{jk,n}^{(l,m)}$ is a *real variable* which depends on both $\theta_{k,n}^{(l,m)}$ and $\theta_{j,n}^{(l,m)}$. Usually, it is assumed that both $r_{j,n}^{(l,m)}$ and $r_{k,n}^{(l,m)}$ are independent of *m*.

As pointed out in Chino (2014b), however, the two terms, $r_{j,n}^{(l,m)}$ and $r_{k,n}^{(l,m)}$ are functions of $\boldsymbol{z}_{j,n}$ and its complex conjugate, $\bar{\boldsymbol{z}}_{j,n}$. This means that $\boldsymbol{z}_{j,n}$ in Eq. (1) is not a *holomorphic function*, since the complex conjugate of $\boldsymbol{z}_{j,n}$ is *not differentiable* in the complex space (e.g., Bak & Newman, 1982). To overcome this difficulty, we may consider the complex state space in this model as a *2p*-dimensional Euclidean space.

Model IV

We have recently considered another type of model in which two terms in Eq. (1), i.e., $\mathbf{g}(\boldsymbol{u}_{j,n})$ and $\boldsymbol{z}_0$, are added, the former being a *control* (e.g., Elaydi, 1999; Ott et al., 1990) and the latter a *complex constant vector* (Chino, 2015b). Here, $\mathbf{g}(\boldsymbol{u}_{j,n})$ is a vector function of a complex vector $\boldsymbol{u}_{j,n}$.

(4) Possible dynamical scenarios of solutions

Figure 1 is a possible scenario of the solution of Model I, in which case $w_{jk} = 0.01(1+i)$, and $w_{kj} = -0.02(1+i)$.

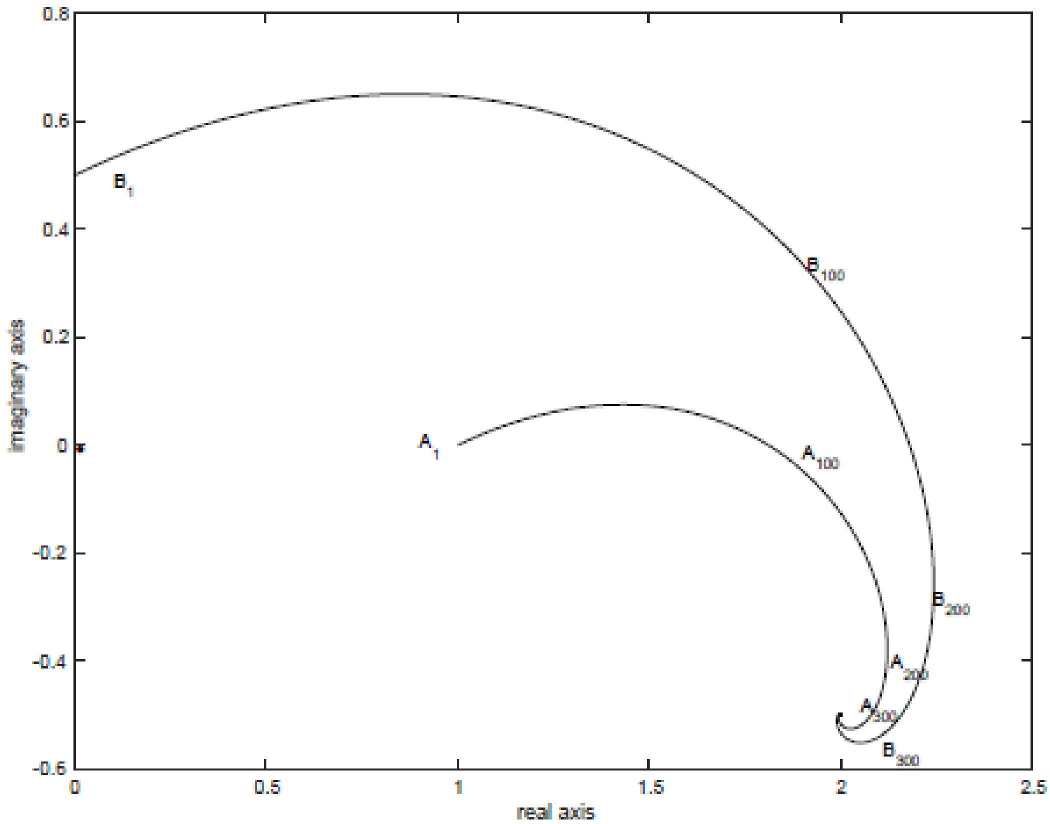

Figure 1. Changes in trajectories of a dyad in a one-dimensional Hilbert space.

It shows the trajectories of a dyad in one-dimensional Hilbert space. Two members A and B located at 1 and 0.5i, respectively, approach to each other gradually, and converge to an equilibrium point. It is not so difficult to prove mathematically that these trajectories converge to a fixed point.

Figure 2 is another possible scenario of the solution of Model I, in which case N=3 (i.e., triadic relation), m=2. In this case, very complicated trajectories are observed.

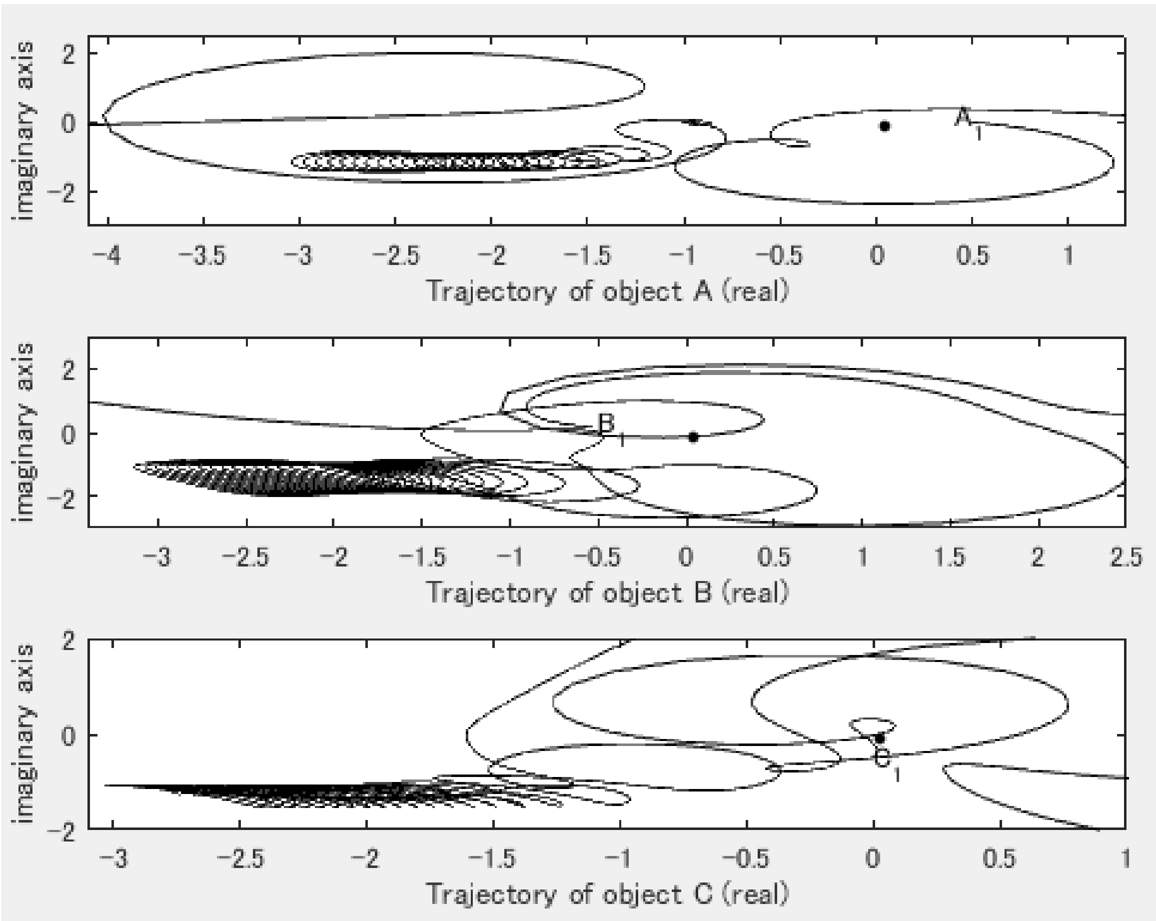

Figure 2. Changes in trajectories of a triad in a one-dimensional Hilbert space.

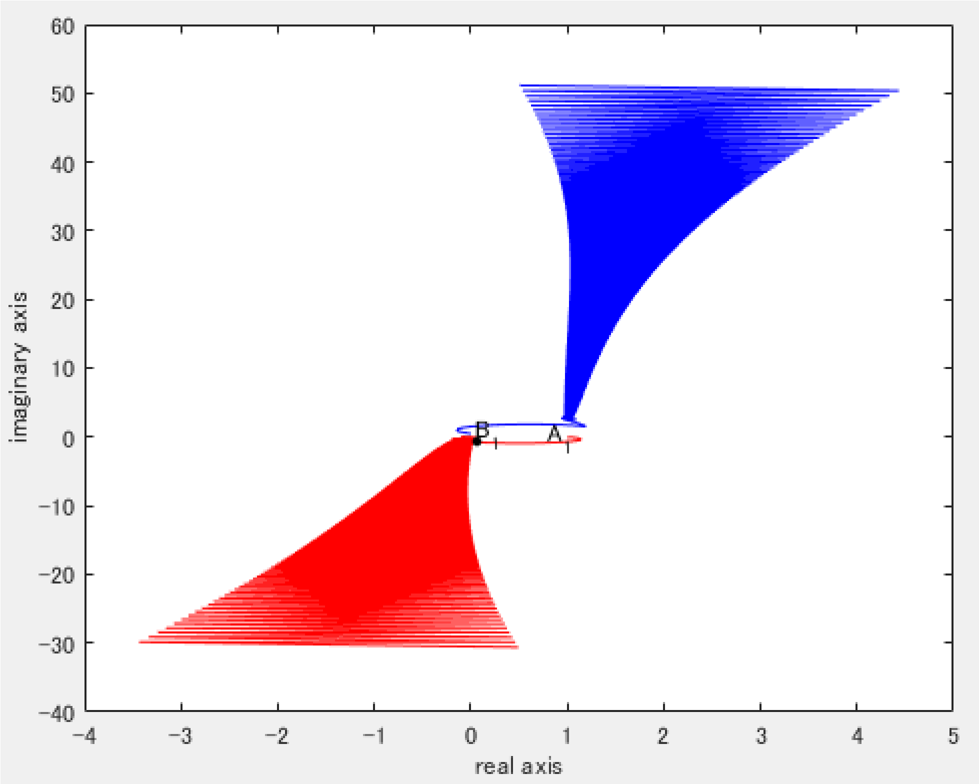

Figure 3. Changes in trajectories of a dyad in a one-dimensional Hilbert space. (Model IV).

Figure 3 is the third possible scenario of the solution of Model IV, in which case N=2 (i.e., dyad relation), m=1, $\mathbf{g}(\boldsymbol{u}_{j,n}) = \mathbf{0}$ and $\boldsymbol{z}_0 = 0.01i$. In this case, both of the two trajectories diverge, as time proceeds.

(5) <u>Relations to dissipative structure and so on</u>

Finally, we shall refer to the relations of our models discussed here to earlier works, mainly to works on *dissipative structure*. Our models may be classified as a *network model* such as the perceptron (e.g., Rosenblatt, 1958; Rumelhart et al., 1986), the recurrent neural network (RNN) (e.g., Hopfield, 1984; Sato, 1990), the automata (specifically, finite automata) (Kleene, 1956), the cellular automaton (e.g., von Neumann & Burks, 1966), the symbolic dynamics (e.g., Hedlund, 1969; Kitchens, 1998), and so on.

Although neural network models such as the perceptron, RNN, and finite automata include *input-output units* in principle, our models do not include them. In this sense, our models are similar to the cellular automaton. Moreover, both our models and the cellular automaton utilize a set of *difference equations*. However, the former is different from the latter in that the former utilizes *(complex) Hilbert space* as the state space of the system, while the latter does not.

In any case, our models enable us to depict various *spatiotemporal structures* of members of any group, which are considered to evolve through *asymmetric interactions* among members. Changes in these spatiotempral structures through such interactions can be thought of as a *self-organizing phenomenon*. In fact, an even simpler system described by one of our models can exhibit, for example, a fixed point behavior (Figure 1), a chaotic behavior, and so on. Then, one might imagine *dissipative structures* to operate in such a phenomenon.

At present we can, at least, discriminate between a *mathematical dissipative system* by Levinson (1944) and the *dissipative structure* by Nicolis and Prigogine (1977). The former is a two-dimensional nonautonomous system which is periodic in t with period L and whose trajectories eventually lies inside of a circle with center at $\boldsymbol{0}$, as time proceeds. In contrast, the latter is attained in an open system far from the equilibrium.

In general, it seems to be not so easy to define such a kind of dissipative structure *in the strict sense* in the phenomena observed in the social and behavioral sciences which we deal with in our model. However, it may be possible, at lease, to define a *computational energy function* originated by Hopfield (1982) and used elsewhere (e.g., Grossberg, 1988; Müezzinoğlu et al., 2003).

References

Bak, J. & Newman, D. J. (1982). *Complex Analysis*, New York:Springer-Verlag.

Chino, N. (2015a). A simulation study of a Hilbert state space model for changes in affinities among memers in informal groups. *Journal of the Institute for Psychological and Physical Science*, **7**, 31-47.

Chino, N. (2015b). Time series analyses of changes in asymmetric relationships among members over time. *Proceeding of the 43th annual meeting of the Behaviormetric Society of Japan* (pp.398-401), Tokyo, Japan.

Chino, N. (2016). Time series analyses of changes in asymmetric relationships among members over time (2). *Proceeding of the 43th annual meeting of the Behaviormetric Society of Japan* , Tokyo, Japan (in print).

Chino, N. & Shiraiwa, K. (1993). Geometrical structures of some non-distance models for asymmetric MDS. *Behaviormetrika*, **20**, 35-47.

Elaydi, S. N. (1996). *An introduction of difference equations*. New York:Springer.

Grossberg, S. (1988). Nonlinear neural networks: Principles, Mechanisms, and Architectures. *Neural Networks*, **1**, 17-61.

Hedlund, G. A. (1969). Endomorphisms and automorphisms of the shift dynamical system. *Mathematical Systems Theory*, **3**, 320-375.

Hopfield, J. M. (1982). Neural networks and physical systems with emergent collective computational abilities. *Proceedings of National Academy of Sciences*, **79**, 2554-2558.

Hopfield, J. J. (1984). Neurons with graded response have collective computational properties like those of two-state neurons. *Proceedings of the National Academy of Sciences*, **81**, 3058-3092.

Kitchens, B. P. (1998). *Symbolic Dynamics*. Berlin: Springer.
Kleene, S. C. (1956). Representation of events in nerve nets and finite automata. In C. E. Shannon & J. McCarthy (Eds.), *Automatic Studies (AM-34)* (pp.3-41). Princeton: Princeton University Press.
Levinson, N. (1944). Transformation theory of non-linear differential equations of the second order. *Annals of Mathematics*, **45**. 723-737.
Müezzinoğlu, M. K., Güzeliş, C., & Zurada, J. M. (2003). A new design method for the complex-valued multistate Hopfield associative memory. IEEE Transactions on Neural Networks, 14, 891-899.
Nicolis, G. & Prigogine, I. (1977). *Self-organization in non-Equilibrium systems –From dissipative structures to order through fluctuations*. New York: Wiley.
Ott, E., Grebogi, C., & Yorke, J. A. (1990). Controlling chaos. *Physical Review Letters*, **64**, 1196-1199.
Rosenblatt, F. (1958). The perceptron: A probabilistic model for information storage and organization in the brain. *Psychological Review*, **65**. 386-408.
Rumelhart, D. E., Hinton, G. E., & Williams, R. J. (1986). Learning representations by back-propagating errors. *Nature*, **323**, 533-536.
Saburi, S. & Chino, N. (2008). A maximum likelihood method for an asymmetric MDS model. *Computational Statistics and Data Analysis*, **52**, 4673-4684.
Sato, M. (1990). A learning algorithm to teach spatio-temporal patterns to recurrent neural networks. *Biological Cybernetics*, **62**, 259-263.
Von Neumann, J. & Burks, A. W. (1966). *Theory of self-reproducing automata*. Illinois: University of Illinois Press.