Hermitian operators, observables, and energy in dynamical asymmetric MDS

Naohito Chino

Dept. of Psychology, Aichi Gakuin University

September 6, 2013

Handout presented at the 41st annual meeting of the Behaviormetric Society of Japan, Chiba.

Organization of today's talk

1. We shall revisit our *complex Hilbert space model* as a dynamical system.

2. We shall discuss the roles of *observables* and *Hermitian operators* in quantum systems.

3. We shall introduce the notion of observables in psychological systems, and examine the differences in observables as well as Hermitian operators between quantum systems and psychological systems.

4. We shall introduce a notion of energy in social and behavioral systems, and discuss the implication of introducing it.

1. Introduction

The acquaintance process seems to be an interesting subject of research in social psychology. For example, Newcomb (1961) observed changes in asymmetric sentiment relationships among members of some informal group over time.

(Digression 1)

The table shown below is his Week 0 data (reproduced from Chino et. al., 2012, p.26).

表 1.11: １７名の成員間の第０週目の対人魅力のランクデータの一部

f\t	1	2	3	4	5	6	7	···	11	12	13	14	15	16	17
1		7	12	11	10	4	13	···	3	9	1	5	8	6	2
2	8		16	1	11	12	2	···	15	6	7	9	5	3	4
3	13	10		7	8	11	9	···	2	1	16	12	4	14	3
4	13	1	15		14	4	3	···	6	9	8	11	10	5	2
5	14	10	11	7		16	12	···	2	3	13	15	8	9	1
6	7	13	11	3	15		10	···	14	5	1	12	9	8	6
7	15	4	11	3	16	8		···	5	2	14	12	13	7	1
8	9	8	16	7	10	1	14	···	2	5	4	15	12	13	6
9	6	16	8	14	13	11	4	···	1	2	9	5	12	10	3
10	2	16	9	14	11	4	3	···	15	8	12	13	1	6	5
11	12	7	4	8	6	14	9	···		2	10	15	11	5	1
12	15	11	2	6	5	14	7	···	3		16	8	9	12	1
13	1	15	16	7	4	2	12	···	6	11		10	3	9	5
14	14	5	8	6	13	9	2	···	12	7	15		4	11	10
15	16	9	4	8	1	13	11	···	3	5	10	15		14	7
16	8	11	15	3	13	16	14	···	2	6	10	7	5		4
17	9	15	10	2	4	11	5	···	8	1	6	16	14	13	

(Nordlie (1958) の Appendix A, Group 2, week 0 を改変したもの)

(end of digression 1)

His data is characterized by a set of two-mode three-way asymmetric relational data matrices. One promising method for analyzing such a data might be to utilize ***dynamical asymmetric MDS models*** **(DAMDS models)**.

Theoretically, we may choose, for example, ***dynamical system model, time series model, or stochastic process model*** **such as stochastic differential equation model**, as candidates for the DAMDS model. Especially, the dynamical system models can be divided further into difference equation models and differential equation models.

There have been several models which can be thought of as versions of DAMDS models. Major ones may be Chino (2003), Chino and Nakagawa (1990), Tobler (1976-1977), and Yadohisa and Niki (1999).

In Tobler's wind model as well as in Yadohisa-Niki model, coordinates of objects are embedded as points in a Euclidean space first using the symmetric part of an asymmetric relationship matrix, and then scalar potentials or vector potentials are estimated using its skew-symmetric part.

On the other hand, in Chino-Nakagawa's differential equation model, coordinates of objects are located in a Euclidean space, and longitudinal vector fields as well as some of the fundamental trajectories associated with each of the vector fields are estimated simultaneously, given a set of longitudinal asymmetric relationship matrices.

By contrast, in **Chino's complex difference equation model** (Chino, 2003), coordinates of members are assumed to be located in a ***finite-dimensional complex Hilbert space***. This model predicts not only the asymmetric relationships between members at any instant of time but also the location of each member at any time in the Hilbert space estimated. As for complex dynamical systems, see elsewhere (i.g., Alexander, 1994; Kravtsov, & Kadtke, 1996).

(Digression 2)

$$\boldsymbol{z}_{j,n+1} = \boldsymbol{z}_{j,n} + \sum_{m=1}^{r} \sum_{k \neq j}^{N} \boldsymbol{D}_{jk,n}^{(m)} \, \boldsymbol{f}^{(m)}\left(\boldsymbol{z}_{k,n} - \boldsymbol{z}_{j,n}\right), \quad \text{j=1,2, ... , N,}$$

where

$$\boldsymbol{f}^{(m)}\left(\boldsymbol{z}_{k,n} - \boldsymbol{z}_{j,n}\right) = \begin{pmatrix} \left(z_{k,n}^{(1)} - z_{j,n}^{(1)}\right)^m \\ \left(z_{k,n}^{(2)} - z_{j,n}^{2}\right)^m \\ \vdots \\ \left(z_{k,n}^{(p)} - z_{j,n}^{(p)}\right)^m \end{pmatrix}. \qquad \text{eq. (1)}$$

Moreover, $D_{jk,n}^{(m)} = \text{diag}\left(w_{jk,n}^{(1,m)}, \dots, w_{jk,n}^{(p,m)}\right)$, and

$$w_{jk,n}^{(l,m)} = a_n^{(l,m)} \, r_{j,n}^{(l,m)} \, r_{k,n}^{(l,m)} \sin\left(\theta_{k,n}^{(l,m)} - \theta_{j,n}^{(l,m)}\right),$$

$$l = 1,2, \dots, p, \;\; m = 1,2, \dots, r. \qquad \text{eq. (2)}$$

Theory and applications of complex dynamical systems in life sciences are few, although those in mathematics have a long history (e.g., Alexander, 1994). However, there has been an increasing attention to them in the neural network literature recently (see, e.g., Aizenberg et al., 1971; Hirose, 2005).

According to the classical Hebb learing rule, the simplest linear dynamical system may be expressed as

$$\tau \frac{d\boldsymbol{J}}{dt} = -\boldsymbol{J} + \boldsymbol{\xi}_{\mu}\boldsymbol{\xi}_{\mu}^{t}, \qquad \mathrm{J} = \{J_{ij}\} = \{J_0 \textstyle\sum_{\mu=1}^{p} \xi_i^{\mu}\xi_j^{\mu}\}, \qquad \text{eq. (3)}$$

and J_{ij} is the synaptic efficacy on neuron *j* of neuron *i*. In this case, matrix $\boldsymbol{J}$ is symmetric.

By contrast, according to the asymmetrically diluted neural networks, matrix $\boldsymbol{J}$ is no longer symmetric. If we further extend the Hebb rule into the complex domain, the complex Hebb rule is obtained. According to Hirose (2005), studies of the complex neural network go back to the early 1970's. One candidate for such a network which he introduced (see, e.g., Chino et al., 2012; Hirose, 2005) is as follows:

$$\tau \frac{d\boldsymbol{H}}{dt} = -\boldsymbol{H} + \boldsymbol{\xi}_{\mu}\boldsymbol{\xi}_{\mu}^{*}, \qquad \text{eq. (4)}$$

where matrix $\boldsymbol{H}$ is an Hermitian matrix whose elements denote some synaptic efficacy, and the symbol '*' denotes conjugate transpose.

(end of digression 2)

In this paper we shall consider roles and natures of DAMDS models, especially models which assume the Hilbert space to be a state space of the dynamical system under study. In section 2 we shall discuss the roles of *observables* and *Hermitian operators* in quantum systems. In section 3 we shall introduce the notion of observables in psychological systems, and examine the differences in observables as well as Hermitian operators between quantum systems and psychological systems. In section 4 we shall discuss the implication of introducing the notion of energy in social systems.

2. Roles of observables and Hermitian operators in quantum systems

The behavior of an elementary particle in the microscopic level cannot be tractable deterministically but can be stochastically. Moreover, a **complex Hilbert space** *H*, is associated with **any quantum system**. Therefore, any ***state* of the quantum system** is described by a vector in the complex Hilbert space.

According to Blank et al. (1994), the influence of the measuring process on the investigated object cannot be made arbitrarily small in quantum physics. Consequently, a state is a result of a sequence of physical manipulations with the system. Considering this point, the notion of an ***observable*** is introduced. A suitable instrument (measuring apparatus) is associated with it, which displays (records) a measured value when we let it interact with the system.

Another important requirement of a mathematical nature in quantum mechanics is that a ***self-adjoint operator*** $\boldsymbol{A}$ (***Hermitian operator***) on the state space is associated with any observable of the system. As shown in Blank et al. (1994), the ***Pauli matrices*** are examples of Hermitian operators, which are expressed as $\boldsymbol{S}_j = \frac{1}{2}\hbar\boldsymbol{\sigma}_j,\ j = 1,2,3$, where $\hbar = \frac{h}{2\pi}$, and h is the Plank constant:

$$\boldsymbol{\sigma}_j = \begin{pmatrix} \delta_{j3} & \delta_{j1} - i\delta_{j2} \\ \delta_{j1} + i\delta_{j2} & -\delta_{j3} \end{pmatrix}, \qquad (1)$$

where δ is the Kronecker symbol. Usually, $\hbar$ is set equal to 1 for simplicity.

(Digression 3)

That is,

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and } \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \text{eq. (5)}$$

(end of digression 3)

The Pauli matrices are examples of observables and describe electron spin states in the Hilbert space C^2, and the spin projections at the j-th axis correspond to the operators $\boldsymbol{S}_j$. The eigenvalues $\pm\frac{1}{2}$ of $\boldsymbol{S}_j$ are the only possible outcome of the measurement.

Finally, in quantum mechanics ***the mean value*** of the measurement results is given by the relation $\langle \boldsymbol{A}\rangle_\psi = (\boldsymbol{\psi},\ \boldsymbol{A}\boldsymbol{\psi})$, which is a ***Hermitian*** ~~***inner product***~~ form. It should be noticed that $\langle \boldsymbol{A}\rangle_\psi$ **is real**, although $\boldsymbol{\psi}$ and $\boldsymbol{A}$ are generally complex.

(Digression 4)

Although the mean value $\langle \boldsymbol{A}\rangle_\psi$ is generally real (but not necessarily positive) like the matrix $\boldsymbol{S}_j$, both of the matrices, $\boldsymbol{E}_j^{(+)}$ and $\boldsymbol{E}_j^{(-)}$, which constitute $\boldsymbol{S}_j$ in such a way that $\boldsymbol{S}_j = \frac{1}{2}\left(\boldsymbol{E}_j^{(+)} - \boldsymbol{E}_j^{(-)}\right)$, are p.s.d. (and the eigenvalues are both 2 and 0. Here,

$$\boldsymbol{E}_j^{(+)} = \frac{1}{2}\left(\boldsymbol{I} + \boldsymbol{\sigma}_j\right), \quad \text{and} \quad \boldsymbol{E}_j^{(-)} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{\sigma}_j\right). \qquad \text{eq. (6)}$$

It is interesting to notice that (simplified versions of) the Pauli matrices, $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3$, form **an orthonormal basis for a unitary space**, and therefore constitute an abstract vector space. In fact, the ***Frobenius inner products***, $(\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k) = tr\left(\boldsymbol{\sigma}_j\, \boldsymbol{\sigma}_k^H\right),\ j \neq k,$ are all zero, and $\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = \boldsymbol{\sigma}_3^2 = \boldsymbol{I}$. For example,

$$(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = tr\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\overline{\begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}}^{t}\right\} = tr\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right\} = 0,$$

eq. (7)

and

$$\boldsymbol{\sigma}_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \boldsymbol{I}.$$ eq. (8)

(end of digression 4)

(Digression 5)

As another example, we shall show the Hermitian matrices of order 4 some of which correspond to the velocity vectors, $\dot{x}/c\,, \dot{y}/c\,, \dot{z}/c,$ of an electron in the magnetic field. Breit (1928) discusses them, which we shall introduce in more detail later in Digression 6. These are Dirac's matrix operators, each of whose eigenvalues are all 1, 1, -1,-1, but the only possible values are known to be 1:

$$\boldsymbol{\alpha}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{\alpha}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\alpha}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$ eq. (9)

(end of digression 5)

3. Introduction to the notion of observable in psychological systems

As reviewed in the introductory section, coordinates of members in psychological systems are assumed to be embedded in a complex Hilbert space in Chino's difference equation model. Of course, we can consider some differential equation model, in which coordinates are assumed to be located in a complex Hilbert space.

The reason is that in both of these models a Hermitian matrix ***H*** is uniquely constructed from each of the one-mode two-way real asymmetric relational data matrices, say, ***S***, which is observed at each point in time, in such a way that

$$\mathbf{H} = \boldsymbol{S}_s + i\,\boldsymbol{S}_{sk} = \frac{1}{2}(\boldsymbol{S} + \boldsymbol{S}^t) + \frac{1}{2}i\,(\boldsymbol{S} - \boldsymbol{S}^t), \qquad (2)$$

Furthermore, in HFM (Chino & Shiraiwa, 1993), a ***Hermitian form*** is associated with h_{jk}, which is the (*j, k*) element of ***H***,

$$h_{jk} = \psi(\boldsymbol{\zeta},\ \boldsymbol{\tau}) = \boldsymbol{\zeta}\,\boldsymbol{\Lambda}\,\boldsymbol{\tau}^{*}, \qquad (3)$$

where $\boldsymbol{\zeta}$ and $\boldsymbol{\tau}$ are row vectors of the matrix whose columns are composed of eigenvectors corresponding to the nonzero eigenvalues of ***H***, and $\boldsymbol{\Lambda}$ is a diagonal matrix with these eigenvalues.

A Hermitian form defined in (3) is said to be Hermitian inner product, if $\psi(\boldsymbol{\zeta},\ \boldsymbol{\zeta}) > 0$ for any $\boldsymbol{\zeta} \neq \mathbf{0}$. Thus, members are embedded in a complex Hilbert space, if ***H*** is p.s.d., which is the result proven by Chino and Shiraiwa (1993).

At this point it seems to be possible to introduce the notion of ***observables in psychological systems*** by analogy with that of quantum systems. In a psychological system in which members interact with each other, a Hermitian operator ***H*** defined by (2) or $\boldsymbol{\Lambda}$ may be said to be an observable in such a psychological system. Neither of them are observed directly. However, we can estimate the eigenvalues of ***H*** or equivalently, matrix $\boldsymbol{\Lambda}$, by measuring an asymmetric relationship matrix ***S*** as counterpart of ***H*** at each point in time.

Of course, there exist a few differences in observables between the psychological system under consideration and quantum systems. On the one hand, Hermitian ~~inner product~~ form must always be real in quantum systems. On the other hand, it is complex in most cases in the psychological system under consideration. The only exceptions are the diagonal entries, i.e., $h_{jj,}$ which are expressed as $\boldsymbol{\zeta \Lambda \zeta^*}$ and are equal to s_{jj}. Such a quantity is sometimes called the ***self-similarity***.

Another important difference between them is in the assumption about the behavior of members or particles. In quantum systems it can be tractable merely **stochastically**. In the psychological system under consideration, we assume at present that it can be tractable **deterministically** up to measurement errors.

4. Introduction of the notion of energy in psychological systems

Another interesting issue may be whether it might be fruitful to introduce **the notion of energy in psychological systems**. In classical physics as well as in quantum physics it is well known that the ***Hamiltonian*** is concerned with the **conservation of the total energy of the system**.

(Digression 6)

As is well known, **Hamiltonian system** is defined as follows (e.g., Perko, 1991):

Definition 1. Let $\boldsymbol{E}$ be an open subset of $\boldsymbol{R}^{2n}$ and let H is an element in $C^2(\boldsymbol{E})$ where $H = \mathrm{H}(\boldsymbol{x},\ \boldsymbol{y})$ with $\boldsymbol{x}, \boldsymbol{y}$ in R^n. A system of the form

$$d\boldsymbol{x}/_{dt} = {}^{\partial H}/_{\partial \boldsymbol{y}}, \qquad {}^{d\boldsymbol{y}}/_{dt} = -\,{}^{\partial H}/_{\partial \boldsymbol{x}}, \qquad \text{eq. (10)}$$

where

$${}^{\partial H}/_{\partial \boldsymbol{x}} = \left({}^{\partial H}/_{\partial \boldsymbol{x}_1},\ \dots,\ {}^{\partial H}/_{\partial \boldsymbol{x}_n}\right)^t,$$

$$\text{and} \quad \partial H / \partial \boldsymbol{y} = \left(\partial H / \partial \boldsymbol{y}_1, \; \ldots, \; \partial H / \partial \boldsymbol{y}_n \right)^t, \qquad \text{eq. (11)}$$

is called a Hamiltonian system with n degrees of freedom on $\boldsymbol{E}$.

It is easy to show that the Hermitian system has an interesting property shown in the following theorem (e.g., Perko, 1991):

Theorem 1 (**conservation of energy**). **The total energy $\mathbf{H}(\boldsymbol{x}, \;\boldsymbol{y})$** of the Hermitian system (11) **remains constant along trajectories**.

(Example 1)

One particular type of Hamiltonian system with one degree of freedom is **the Newtonian system with one degree of freedom** (e.g., Perko, 1991),

$$d^2x / d\,t^2 = \mathrm{f}\,(x), \qquad \text{eq. (12)}$$

where function f is an element in $C^1(a, b)$. This differential equation can be rewritten as a system in R^2:

$$dx / dt = \mathrm{y}, \qquad dy / dt = f(x). \qquad \text{eq. (13)}$$

The total energy for this system can be written as

$$\mathrm{H(x, y) = U(x) + T(y)}, \qquad \text{eq. (14)}$$

where

$$\mathrm{U}(x) = -\int_{x_0}^{x} f(s)\, ds, \quad \text{and} \;\; \mathrm{T}(y) = y^2/2, \qquad \text{eq. (15)}$$

which are **the potential energy** and **the kinetic energy**, respectively.

(Example 2)

The classical Hamiltonian function $\boldsymbol{H}$ for an electron having a charge $(-e)$ is related to the momenta p_x, p_y, p_z by the equation (Breit, 1928),

$$H/c + e\,A_0/c$$

$$= \sqrt{m^2c^2 + \left(p_x + \frac{eA_x}{c}\right)^2 + \left(p_y + \frac{eA_y}{c}\right)^2 + \left(p_z + \frac{eA_z}{c}\right)^2}, \qquad \text{eq. (16)}$$

where $A_0, (A_x,\ A_y,\ A_z)$ are, respectively, the electrostatic and the vector potentials of the field.

According to Breit (1928), if we rewrite eq. (16) in another way and comparing it with Dirac's equation,

$$\left\{p_0 + \frac{eA_0}{c} + \alpha^{(1)}\left(p_x + \frac{eA_x}{c}\right) + \alpha^{(2)}\left(p_y + \frac{eA_y}{c}\right) + \alpha^{(3)}\left(p_z + \frac{eA_z}{c}\right) + \alpha^{(4)}\,mc\right\}\psi = 0,$$

eq. (17)

we have

$$p_0 = \frac{H}{c}, \alpha^{(1)} = -\frac{\dot{x}}{c},\ \alpha^{(2)} = -\frac{\dot{y}}{c},\ \alpha^{(3)} = -\frac{\dot{z}}{c},\ \alpha^{(4)} = -\sqrt{1-\beta^2}.$$

eq. (18)

The Hamiltonian matrices, $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$ shown in digression 5 are operational representations of $-(\dot{x}/c\,, \dot{y}/c\,, \dot{z}/c)$ (Breit, 1928).

(end of digression 6)

Is it possible to find an appropriate Hamiltonian function in the psychological system under study? At present, such a problem seems to be an open question. A further curious question would be to ask whether psychological systems are ***conservative*** *or* ***dissipative*** (i.g., Iooss & Joseph, 1990; Nicolis & Prigogine, 1977).

(Digression 7)

A simpler conservative system is known as **the conservative force field** in physics. In general a vector field $F: R^3 \to R$ is called **a force field** if **the vector $\boldsymbol{F}(\boldsymbol{x})$** assigned to the point $\boldsymbol{x}$ is interpreted as **a force** acting on a particle placed at $\boldsymbol{x}$.

Conservative force fields are defined as follows (see, e.g., Hirsch & Smale, 1974):

Definition 2. Let V be a C^1 function

$$\mathrm{V}: R^3 \to R$$

such that

$$\boldsymbol{F}(\boldsymbol{x}) = -(\partial V(\boldsymbol{x})/\partial x_1, \partial V(\boldsymbol{x})/\partial x_2, \partial V(\boldsymbol{x})/\partial x_3)$$

$$= -\mathrm{grad}\, V(\boldsymbol{x}). \qquad \text{eq. (19)}$$

Such a force field is called conservative. Here, the function V is called **the potential energy** function.

(Example 1)

For **a particle moving in a conservative field** $\boldsymbol{F}(\boldsymbol{x}) = -\mathrm{grad}\,\mathrm{V}$, the potential energy is $V(\boldsymbol{x})$, while the kinetic energy is $\frac{1}{2}\,|m\,\dot{\boldsymbol{x}}|^2$. The total energy is $E = V + T$. To be precise, if $\boldsymbol{x}\,(\mathrm{t})$ is the trajectory of a particle moving in the conservative force field, then

$$\mathrm{E(t)} = \mathrm{V}\big(\boldsymbol{x}(t)\big) + \frac{1}{2}\,\big|m\,\boldsymbol{x}\dot{(t)}\big|^2. \qquad \text{eq. (20)}$$

Theorem 2 (**conservation of energy**). Let $\boldsymbol{x}(\mathrm{t})$ be the trajectory of a particle moving in a conservative force field $\boldsymbol{F} = -\mathrm{grad}\,\mathrm{V}$. Then the total energy E is independent of time.

Systems which are, at a glance, similar to conservative force fields but are essentially different systems are **gradient systems**. Gradient systems are defined as follows (see also, Hirsch & Smale, 1974):

Definition 3. A gradient system on an open set $W \subset R^n$ is a dynamical system of the form

$$\mathbf{dx}/_{dt} = -\text{grad V}, \qquad \text{eq. (21)}$$

where $V: U \to R$ is a C^2 function, and

$$\text{grad } V = (\partial V/\partial x_1, \partial V/\partial x_2, \partial V/\partial x_3). \qquad \text{eq. (22)}$$

is **the gradient vector field** grad V: $U \to R^n$ of V.

(Example 1)

Level curves of **the contour map** depicted in Abelson (1955) as well as Kosugi and Kawatani (2012) are, at a glance, those of the potential energy in a gradient system or those in a conservative force field. Although their definitions of the vector fields are empirical, neither of them is not so precise as those in mathematics.

(end of digression 7)

(Digression 8)

We have seen that there exists some Hamiltonian function in the Newtonian system as well as the quantum system. In such systems **the Hamiltonian total energy is conserved**. However, it seems likely that the total energy might not be conserved sometimes in systems in life sciences as well as social and behavioral sciences.

At this point some natural questions may arise. Are there any conservative systems in a broader sense than those in the Hamiltonian system? The answer is yes. For example, Lorenz (1993) refers to the following difference equation, which is some type of the logistic map,

$$x_{n+1} = \mathrm{a}\, x_n(1 - x_n). \qquad \text{eq. (23)}$$

If we introduce new parameters, c and v, such that

$$\mathrm{c} = a/2 - a^2/4, \quad \text{and } \ v_n = \mathrm{a}(1 - 2x_n)/2,$$

then we have

$$v_{n+1} = v_n^2 + c. \qquad \text{eq. (24)}$$

Eq. (24) is equivalent to

$$v_{n+1} = v_n + v_n^2 - w_n^2, \qquad w_{n+1} = v_n. \qquad \text{eq. (25)}$$

Lorenz points out that the quantity, $\boldsymbol{v - w^2}$, in Eq. (21) is preserved.

Apart from the conservative system discussed above, some energy functions have been defined in both real and complex associative memory neural networks (e.g., Hirose, 2005). As for the latter case, the energy function is defined as

$$\mathrm{E}(\boldsymbol{y}; \boldsymbol{H}) = -\boldsymbol{y}^* \boldsymbol{H}\, \mathbf{y}, \qquad \text{eq. (26)}$$

where $\boldsymbol{H}$ is the Hermitian matrix defined in Eq. (4) in digression 2, and $\boldsymbol{y}$ is a column vector which denotes any state of the system under study. This function is composed of a Hermitian form and reminiscent of the mean value of measurement in quantum mechanics.

This function might also be a possible and promising energy function to be defined in social and behavioral systems, an example of which is the acquaintance process considered by Newcomb (1961).

(end of digression 8)

Additional references

Abelson,R. P. (1955). A technique and a model for multi-dimensional attitude scaling. *Public Opinion Quarterly*, **18**, 405-418.

Breit, G. (1928). An interpretation of Dirac's theory of the electron. *Proceedings of the National Academy of Sciences, U. S. A.*, **14**, 553-559.

Chino, N., Saburi, S., & Okada, K. (2012). *Hi-taisho MDS no Riron to Ohyo* [*Theory and Applications of Asymmetric MDS*]. Kyoto: Gendai Sugaku.

Hirose, A. (2005). Fukuso Nyuralu Nettowahku [Complex Neural Network]. Tokyo: Saiensu Co.

Hirsch, M. W., & Smale, S. (1974). Differential equations, dynamical systems, and linear algebra. New York: Academic Press.

Kosugi, K., & Kawatani, Y. (2012). Hitaisyo-deita no sanjigen kukan moderu [Three-dimensional spatial model for asymmetric data]. *Proceedings of the 40th annual meeting of the Behaviormetric Society of Japan*. 279-280.

Lorenz, E. N. (1993). *The Essence of Chaos*. Washington: University of Washington Press.

Perko, L. (1991). *Differential Equations and Dynamical Systems*. New York: Springer.